# Microphysics of $SO(10)$ Cosmic Strings

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## Abstract

We uncover a rich microphysical structure for $SO(10)$ cosmic strings. For the abelian string the electroweak symmetry is restored around it in a region depending on the electroweak scale. Four distinct nonabelian strings are found. Some of these also restore the electroweak symmetry. We investigate the zero mode structure of our strings. Whilst there are right handed neutrino zero modes for the abelian string, they do not survive the electroweak phase transition in the case of the lowest energy solution. We elucidate the zero mode structure for the nonabelian strings above and below the electroweak phase transition. We consider the generalisation of our results to other theories and consider the cosmological consequences of them.

# 1   Introduction

Topological defects arise as a result of phase transitions in the early universe [1]. Such topological defects, in particular cosmic strings, resulting from a grand unified theory (GUT), generate density perturbations which could explain the observed large-scale structure and anisotropy in the microwave background, see [1] and the references therein. They also provide an important window into the physics of the very early universe. For example, in the core of the cosmic strings the underlying GUT symmetry is restored, resulting in baryon violating processes being unsuppressed. This can catalyse proton decay [2], and the decay of string loops can explain the observed baryon asymmetry [3].

In recent years it has become increasingly apparent that cosmic strings have a richer microstructure than previously thought [4, 5]. In particular, at subsequent phase transitions the core of the cosmic string acquires additional features. For example, the string can cause electroweak symmetry restoration in a much larger region around it, proportional to the electroweak scale itself [4, 5]. This new microphysical structure has been used to provide a new scenario for electroweak baryogenesis [6], and to investigate the current-carrying properties of cosmic strings [7, 8].

Previous work considered the simplest extension to the Standard Model that would allow the formation of strings. A $U(1)$ symmetry, whose breaking produced an abelian string, was added to the usual Standard Model symmetries. The resulting theory had two coupling constants, of arbitrary ratio. It was shown that if the ratio was large enough, the electroweak Higgs field would not only be zero at its centre, but would also wind like a string. Whether this is likely to happen with phenomological strings can be found by considering a realistic grand unified theory, where there is less arbitrariness.

By using a larger gauge group it is also possible to consider the effects of nonabelian strings, which could not occur in the theories considered in [4, 5]. Nonabelian strings have significantly different behaviour to abelian strings, since the associated string generators do not all commute with the Standard Model fields, or the other gauge fields. It is thus necessary to approach them in a slightly different way.

As well as symmetry restoration, the presence of a string may allow the formation of non-trivial zero energy fermion solutions, as discussed by Jackiw and Rossi [9], for an abelian string. If such zero modes exist, a superconducting current can flow along the string which may have long range effects [11]. It is possible that the electroweak Higgs field, because of its string-like form, could also allow the formation of such zero modes [8, 10, 11].

In this paper we examine these issues in detail for strings formed in a realistic grand unified theory based on $SO(10)$. In section 2 the grand unified theory to be used is outlined. The possible strings that form in it prior to the electroweak phase transition are discussed in section 3. In section 4 the effect of the strings on the electroweak symmetry are considered, and the approximate form of the

fields, as well as an estimate of the energy is found. In section 5 some other, simpler symmetry restorations occuring in the theory are discussed, in particular that of the intermediate $SU(5)$ symmetry. In section 6 the possible existence of fermion zero modes for abelian and nonabelian strings is investigated, including cases in which two Higgs fields effect the fermion fields. Although one specific theory is considered, many of the results generalise to other theories. The implications of our results for such theories are discussed in section 7. In section 8 we summarise our results and discuss the conclusions.

## 2  An $SO(10)$ Grand Unified Theory

A realistic GUT which has a symmetry breaking pattern which produces strings is $SO(10)$. Its properties have a reasonable agreement with physical results. Consider the symmetry breaking

$$
\begin{aligned}
SO(10) \quad &\overset{\Phi_{126}}{\longrightarrow} \quad SU(5) \times Z_2 \\
&\overset{\Phi_{45}}{\longrightarrow} \quad SU(3)_c \times SU(2)_L \times U(1)_Y \times Z_2 \\
&\overset{\Phi_{10}}{\longrightarrow} \quad SU(3)_c \times U(1)_Q \times Z_2
\end{aligned}
\tag{1}
$$

where $\Phi_N$ transforms under the $\mathbf{N}$ representation of $SO(10)$. The actual grand unified gauge group is $Spin(10)$, the covering group of $SO(10)$, but for simplicity the symmetry breaking is shown in terms of the Lie algebras. The discrete $Z_2$ symmetry formed by the $\Phi_{126}$ Higgs field leads to the formation of a variety of cosmic strings. Comparison of the effects of the various symmetry breakings is simplified by expressing everything in terms of the same representation. Since $\mathbf{126} + \mathbf{10} = (\mathbf{16} \times \mathbf{16})_S$ this is possible. Conveniently, $\mathbf{16}$ is also the representation that acts on the fermions. The $SO(10)$ fermions consist of the usual standard model fermions, plus a right handed neutrino. The fermionic part of the theory is then expressed in terms of the left-handed fermions and the charge conjugates of the right-handed fermions.

The maximal subgroup of $SO(10)$ is actually $SU(5) \times U(1)_P$, and $P$ can be used to decompose $SO(10)$ into representations of $SU(5)$

$$
\mathbf{16} \longrightarrow \mathbf{1}_5 + \mathbf{10}_1 + \bar{\mathbf{5}}_{-3}
\tag{2}
$$

where the subscripts are the eigenvalues of $P$. $\mathbf{126}$ and $\mathbf{10}$ can be similarly decomposed by considering symmetric products of $\mathbf{16}$.

$$
\mathbf{126} \longrightarrow \mathbf{1}_{10} + \ldots \qquad\qquad \mathbf{10} \longrightarrow \mathbf{5}_{-2} + \bar{\mathbf{5}}_2
\tag{3}
$$

$P$ can also be used to describe the non-trivial element of the discrete symmetry of (1), which is $D = \exp(2\pi i P/10)$.

The vacuum expectation value of $\Phi_{126}$ has a magnitude of $\frac{\eta_1}{\sqrt{2}}$, which is of order $10^{15}$ GeV. It is in the $\mathbf{1}_{10}$ component of $\mathbf{126}$, and so must be equal to

3

$\frac{\eta_1}{\sqrt{2}}(u \times u)$, where $u$ is in the $\mathbf{1}_5$ component of the $\mathbf{16}$ representation of $SU(5)$ (the corresponding field in the fermion representation is the charge conjugate of the right-handed neutrino). In $SU(5)$ the VEV of the equivalent of $\Phi_{10}$ is in the chargeless component of $\mathbf{5}_{-2}$. This choice can create problems in $SO(10)$ (see section 6.2.4), so the VEV of $\Phi_{10}$ is made up of both the chargeless components of $\mathbf{10}$ instead. If $H^0$ and $\tilde{H}^0$ are the chargeless components of $\mathbf{5}_{-2}$ and $\bar{\mathbf{5}}_2$ respectively, $\Phi_{10}$ can be expressed as $\frac{\eta_2}{\sqrt{2}}\mathcal{H}^0$, where $\mathcal{H}^0 = H^0\kappa + \tilde{H}^0\tilde{\kappa}$. If the fermion field phases are defined appropriately, $\kappa$ and $\tilde{\kappa}$ will be real and positive, and $\kappa^2 + \tilde{\kappa}^2 = 1$. Since $\mathbf{10}$ is contained in $(\mathbf{16} \times \mathbf{16})_S$, $\mathcal{H}^0$ can be expressed as a sum of symmetric products of components of $\mathbf{16}$s. The $\mathbf{45}$ is contained in $\mathbf{16} \times \bar{\mathbf{16}}$. $\Phi_{45}$'s effect on the formation of strings is far less significant than the other Higgs fields, so it can be ignored for now.

Strings can form at the first $SO(10) \longrightarrow SU(5) \times Z_2$ symmetry breaking. In this case $\Phi_{126}$ is not constant, and takes the form $e^{i\theta T_s}\phi_{126}^{(0)}(r)$. $\phi_{126}^{(0)}$ is independent of $\theta$, and satisfies the boundary condition $\phi_{126}^{(0)}(\infty) = \frac{\eta_1}{\sqrt{2}}u \times u$. $T_s$ is made up of the broken generators of $SO(10)$, and must give a single-valued $\Phi_{126}$. If $e^{2\pi i T_s} = U \times D$ (where $U$ is in $SU(5)$), then while $\Phi_{126}$ will be single valued, it will not be topologically equivalent to $\Phi_{126} = $ constant, and so the string will be topologically stable. If $e^{2\pi i T_s} = U \times I$ the string is not topologically stable, but may have a very long lifetime, and so still be physically significant [11].

The Lagrangian of the system is

$$\mathcal{L} = (D_\mu\Phi_{126})^*(D^\mu\Phi_{126}) + (D_\mu\Phi_{10})^*(D^\mu\Phi_{10}) + (D_\mu\Phi_{45})^*(D^\mu\Phi_{45})$$
$$- \tfrac{1}{4}F_{\mu\nu}^a F^{\mu\nu a} - V(\Phi_{126}, \Phi_{45}, \Phi_{10}) + \mathcal{L}_{\text{fermions}} \tag{4}$$

where $D_\mu = \partial_\mu - \frac{1}{2}igA_\mu$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \frac{1}{2}ig[A_\mu, A_\nu]$. There are 45 gauge fields in all, most of which acquire superheavy masses and so are not observed at everyday temperatures. They consist of the usual standard model fields, with the $W$-bosons denoted by $W_L^i$; $W_R^i$, which are right-handed versions of the $W_L^i$, coupling right handed neutrinos to electrons; some leptoquark bosons: $Y_i^\pm$, $X_i^\pm$, $X_{Si}^\pm$, where $Y_i^\pm$ and $X_i^\pm$ are $SU(5)$ gauge fields; some more general gauge fields $X_i'^\pm$ and $Y_i'^\pm$, which couple quarks to leptons and different coloured quarks; and a fifth diagonal field, $B'$. The index $i$ takes the values 1,2,3, and is related to colour. Two uncharged diagonal fields, $S$ and $B$, are made up of orthogonal linear combinations of $W_R^3$ and $B'$. Linear combinations of $B$ and $W_L^3$ produce the $Z$ boson and the photon, $A$. At the first symmetry breaking $S$, $W_R^\pm$, $X_i'^\pm$, $Y_i'^\pm$, and $X_{Si}^\pm$ are all given superheavy masses. The second stage gives high masses to $X_i^\pm$, $Y_i^\pm$, and additional masses to $W_R^\pm$, $X_i'^\pm$, $Y_i'^\pm$ and $X_{Si}^\pm$. Finally the third symmetry breaking gives masses to $W_L^\pm$ and $Z$, with further masses being given to the $S$, $W_R^\pm$, $Y_i^\pm$ and $Y_i'^\pm$ fields.

## 3  GUT Strings

Neglecting fermions, the Euler-Lagrange equations obtained from (4) are

$$D_\mu D^\mu \Phi_i = -\frac{\partial V}{\partial \Phi_i^*} \tag{5}$$

$$\partial_\mu F^{\mu\nu a} - \frac{1}{2} i g f^c{}_{ab} A_\mu^b F^{\mu\nu c} = -g \mathrm{Im} \sum_i (D^\nu \Phi_i)^* (\tau^a \Phi_i) \tag{6}$$

At high temperatures $V$ is such that $\Phi_{126}$ is the only non-zero Higgs field. (5) and (6) have various string solutions. The different solutions correspond to different choices of $T_s$. In the $SO(10) \longrightarrow SU(5) \times Z_2$ symmetry breaking, 21 of $SO(10)$'s 45 generators are broken, and $T_s$ is a linear combination of them. One of them, $P$, corresponds to the $U(1)$ symmetry not embedded in $SU(5)$. The corresponding string is abelian, and has the solution

$$\begin{aligned}
\Phi_{126} &= f(r) e^{in\theta T_s} \phi_{126}^{(0)}(\infty) \\
A_\theta &= n \frac{2a(r)}{gr} T_s \qquad\qquad A_\mu = 0 \;\; \text{otherwise}
\end{aligned} \tag{7}$$

where $T_s$, the string generator, equals $\frac{P}{10}$, and $n$ is an integer. The non-zero gauge field is required to give a zero covariant derivative, and hence zero energy, at infinity. It corresponds to a non-zero $S$ field. (7) can be simplified using $Pu = 5u$, to give $e^{in\theta T_s} \phi_{126}^{(0)} = e^{in\theta} \phi_{126}^{(0)}$. Substituting (7) into (5) and (6) gives the Nielsen-Olesen vortex equations, as would be expected. Regularity at the centre of the string, and finite energy due to a vanishing covariant derivative and potential at infinity, imply the boundary conditions $f(0) = a(0) = 0$ and $f(\infty) = a(\infty) = 1$.

The situation for the other generators is more complicated. For a general string generator $T_s$, the left and right hand sides of (5) are proportional to $T_s^2 \phi_{126}^{(0)}$ and $\phi_{126}^{(0)}$ respectively, which in general are not proportional [13]. Thus the solution (7) will not work. This is resolved by expressing $\phi_{126}^{(0)}(\infty)$ in terms of the eigenstates of $T_s^2$, to give $\phi_{126}^{(0)}(\infty) = \sum_m \phi_m$, where $T_s^2 \phi_m = m^2 \phi_m$. Since $T_s$ is Hermitian, $m^2$ will be positive and real. A suitable string solution can now be constructed

$$\begin{aligned}
\Phi_{126} &= e^{i\theta T_s} \sum_m f_m(r) \phi_m \\
A_\theta &= \frac{2a(r)}{gr} T_s \qquad\qquad A_\mu = 0 \;\; \text{otherwise}
\end{aligned} \tag{8}$$

In order for $\Phi_{126}$ to be single valued the various $m$ must all be integers, and $T_s \phi_0$ must be zero. The boundary conditions on $a$ and $f_m$ will be the same as those for (7), except that $f_0$ need not be zero at $r = 0$. The simplest examples of such solutions occur when $T_s^2 u = \frac{n^2}{4} u$ (where $n$ is an integer), in which case

$$\phi_n = \frac{\eta_1}{\sqrt{2}} \left( \frac{1}{2} u \times u + \frac{2}{n^2} T_s u \times T_s u \right) \;\text{and}\; \phi_0 = \frac{\eta_1}{\sqrt{2}} \left( \frac{1}{2} u \times u - \frac{2}{n^2} T_s u \times T_s u \right) \tag{9}$$

$T_s u$ and $u$ are orthogonal, so $\phi_0$ and $\phi_n$ are orthogonal. In this case only part of the Higgs field winds around the string. This type of string was first suggested by Aryal and Everett [12], and has been examined in detail by Ma [13]. It turns out to have lower (about half as much) energy than the abelian string (7). This is because the Higgs field is not forced to be zero at the string's centre, which reduces the contribution to the energy from the potential terms. Also since $\Phi_{126}$ varies less, the covariant derivative terms are smaller.

Of course, such vortex-like solutions are only topological strings if $e^{2\pi i T_s}$ is not contained in $SU(5)$. If $n$ is even this is not the case, and the solution is topologically equivalent to the vacuum. Similarly, odd values of $n$ are all topologically equivalent to each other, so there is only one topologically distinct type of string of this form. Strings with higher $n$ can unwind into strings with lower $n$. The same is true of the abelian string. However it is possible that the lifetime of an $n > 1$ string will be very long, so in a general theory all values of $n$ should be considered. Putting $T_s \longrightarrow n T_s$ in (8) and $n = 1$ in (9) gives a similar ansatz to (7), making comparison easier.

As shown in [13] the most general potential reduces to a different form to that of the abelian case, and leads to these equations for $a$ and the $f_m$'s

$$f_0'' + \frac{f_0'}{r} = \eta_1^2 \left[ \lambda_1 \left( \frac{f_1^2 + f_0^2}{2} - 1 \right) - \mu_1 \left( \frac{f_1^2 - f_0^2}{2} \right) \right] f_0 \qquad (10)$$

$$f_1'' + \frac{f_1'}{r} - n^2 \frac{(1-a)^2}{r^2} f_1 = \eta_1^2 \left[ \lambda_1 \left( \frac{f_1^2 + f_0^2}{2} - 1 \right) + \mu_1 \left( \frac{f_1^2 - f_0^2}{2} \right) \right] f_1 \quad (11)$$

$$a'' - \frac{a'}{r} = -\frac{1}{2} g^2 \eta_1^2 (1-a) f_1^2 \qquad (12)$$

where $\mu_1$ and $\lambda_1$ are such that $f_1(\infty)$ and $f_0(\infty)$ will both be 1. The corresponding equations for the abelian string $(T_s = P/10)$ are

$$f'' + \frac{f'}{r} - n^2 \frac{(1-a)^2}{r^2} f = \eta_1^2 \lambda_1 (f^2 - 1) f \qquad (13)$$

$$a'' - \frac{a'}{r} = -\frac{5}{2} g^2 \eta_1^2 (1-a) f^2 \qquad (14)$$

The above nonabelian strings are in fact all $SU(2)$ strings. There are other more complicated possibilities, for which $T_s^2 u$ is not proportional to $u$, but none of these are topologically stable. We shall only consider topologically stable strings, and the closely related solutions with higher winding numbers. If a single valued charge operator is also required there are just a few possibilities for $T_s$, which can be classified in terms of the non-zero gauge fields around the string. These are all equivalent under $SU(5)$, but not under $SU(3)_c \times U(1)_Q$, so they will be distinct after the electroweak symmetry breaking. Apart from the abelian string the four cases correspond to non-zero $W_R^\pm$, $X_{Si}^\pm$, $X_i'^\pm$ and $Y_i'^\pm$ fields. Under $SU(3)_c \times U(1)_Q$ any linear combination of $X_{Si}^\pm$ generators

can be gauge transformed into any other combination, thus they are equivalent. The same is true for the other generators, so there are just 5 distinct types of string at low temperatures. The 4 nonabelian strings can be labelled by their gauge fields. Under $U(1)_Q$, nonabelian strings with winding number $-n$ are gauge equivalent to ones with winding number $n$ (for any choice of $T_s$), so it is sufficient to consider only $n > 0$ strings.

Since all the nonabelian strings are gauge equivalent under $SU(5)$, they have the same energy, which is about half that of the abelian string [13]. Although these strings give single valued electric charge, they do not all give single valued colour charge, and so some are Alice strings. This does not mean that they are unphysical, as discussed in [14].

# 4   The Electroweak Symmetry Breaking

Topological strings only form in a symmetry breaking $G \longrightarrow H$ if $\pi_1(G/H) \neq I$. This is not the case at the third ($\Phi_{10}$) symmetry breaking, so such strings do not form there. It is still possible for $\Phi_{10}$ to wind, and for string-like solutions to appear [15]. However, since it is energetically favourable for the Higgs field to unwind, they are not completely stable (although they could take a long time to decay).

The situation is different in the presence of a topological string, formed at a previous symmetry breaking. If the electroweak Higgs field took its usual constant vacuum expectation value now, and the gauge fields from the string did not annihilate it, its covariant derivative would be non-zero everywhere. At large $r$ it would be proportional to $1/r$, and so it would have logarithmically divergent energy. This can be avoided by allowing $\Phi_{10}$ to wind like the GUT string. In order for it to be single valued, it may be necessary to alter the string generator, $T_s$. Alternatively it may be possible cancel the effect of the string gauge fields by adding other gauge fields which have an opposite effect on $\Phi_{10}$. The most energetically favourable solution is likely to be a combination of these two alterations. Unlike the electroweak strings considered by Vachaspati [15], the resulting string-like solutions would be stable.

## 4.1   The Abelian $U(1)$ String

For the abelian string, the covariant derivative can be made to vanish by letting $\Phi_{10}$ wind. But $T_s H^0 = -\frac{1}{5} H^0$, and $T_s \tilde{H}^0 = \frac{1}{5} \tilde{H}^0$, so $e^{2\pi i n T_s} \phi_{10}^{(0)} \neq \phi_{10}^{(0)}$ in general. When $n$ is not a multiple of 5, just allowing $\Phi_{10}$ to wind is insufficient, since this will give a multi-valued $\Phi_{10}$. Instead $T_s$ must be replaced by another generator, $\tilde{T}_s$, which will give a single valued $\Phi_{10}$. The gauge field must also be changed to give a vanishing covariant derivative. This alteration must not affect the GUT string, as any change would give a large increase in energy. Thus $\tilde{T}_s$ needs to be of the form $T_s + R$, where $R$ annihilates $u$ and $T_s u$, but has some

7

affect on $H^0$. If $RH^0 = kH^0$ and $R\tilde{H}^0 = -k\tilde{H}^0$, ($k$ is real since $R$ is Hermitian), a suitable ansatz is

$$
\begin{aligned}
\Phi_{10} &= \tfrac{\eta_2}{\sqrt{2}}h(r)e^{in(T_s+R)\theta}\mathcal{H}^0 = \tfrac{\eta_2}{\sqrt{2}}h(r)\left(e^{im\theta}H^0\kappa + e^{-im\theta}\tilde{H}^0\tilde{\kappa}\right) \\
A_\theta &= n\tfrac{2}{g}\left(\tfrac{a(r)}{r}T_s + \tfrac{b(r)}{r}R\right) \qquad\qquad A_\mu = 0 \;\; \text{otherwise}
\end{aligned}
\tag{15}
$$

where $m = n\left(k - \tfrac{1}{5}\right)$, and $a(r)$ is determined by the GUT string. Now the covariant derivative of $\Phi_{10}$ does vanish at infinity (provided $b(\infty) = 1$), and if $k$ is such that $m$ is an integer, $\Phi_{10}$ will still be single-valued. Such an $R$ does exist in $SO(10)$, and it is proportional to the generator of $Z$. Strictly, the variation of $\Phi_{10}$ will effect the GUT string, but the effect is of order $\eta_2^2/\eta_1^2 \sim 10^{-26}$, and so can be ignored. ($\Phi_{10}$ takes the role of the Weinberg-Salam Higgs field and so its usual vacuum expectation value, $\tfrac{\eta_2}{\sqrt{2}}$, is of order $10^2$ GeV.) The functions $h(r)$ and $b(r)$ obey the same boundary conditions as $f(r)$ and $a(r)$ in (13,14), for the same reasons, except when $m = 0$, in which case $h(0)$ need not be zero. This gives the field equations

$$
h'' + \frac{h'}{r} - n^2\frac{\left[k(1-b) - \tfrac{1}{5}(1-a)\right]^2}{r^2}h = \eta_2^2\lambda_2(h^2 - 1)h
\tag{16}
$$

$$
b'' - \frac{b'}{r} = -\frac{2}{5}g^2\eta_2^2\left\{(1-b) - \frac{1}{5k}(1-a)\right\}h^2
\tag{17}
$$

Strictly they should include contributions arising from $|\Phi_{126}|^2|\Phi_{10}|^2$ terms in the Lagrangian. However, outside the string core $|\Phi_{126}|$ is constant, and the cross terms can be absorbed into the constant potential terms. Inside the string this is not so, but the effect of the gauge terms on $\Phi_{10}$ far outweighs that of the potential terms, as noted in [5]. Thus $|\Phi_{126}|$ can be taken as being constant everywhere. These are similar in form to the Nielsen-Olesen vortex equations, except for the extra gauge term. They are also similar in that they cannot generally be analytically solved. However, using a similar approach to [4] it is possible to obtain an approximate trial solution, and use it to estimate the energy of (15).

Defining $r_s$ to be the radius of the string (outside of which $|\Phi_{126}|$ takes its usual VEV), and $r_{ew}$ to be the radius of the region in which $|\Phi_{10}|$ does not take its usual VEV, the solution can be approximated separately in three regions. If $g$, $\lambda_1$ and $\lambda_2$ are of order 1, $r_s$ will be of order $\eta_1^{-1}$ and $r_{ew}$ will be of order $\eta_2^{-1}$. At short distances the potential, and additional gauge term, $b(r)$, can be neglected in (16), as the other terms dominate. Inside the string itself, $(r < r_s)$ $a(r) \sim r^2$ can also be ignored giving $h(r) = Cr^{|m|}$ (the other solution is not used since $h(0)$ must be zero). $b(r)$ is made up of two terms, an $r^{2|m|+2}$ term, proportional to $C$, and an arbitrary $r^2$ term. For $r_s < r < r_{ew}$, $a(r)$ is approximately 1, and so $h = Ar^{|nk|} + Br^{-|nk|}$. Requiring continuity of $h$ and $h'$ at $r = r_s$, ensures that $B \sim Ar_s^{2|nk|}$, and so it can be neglected

away from the string. The situation is slightly different for $m = 0$. In this case $h(r) = C$ for $r < r_s$ (the second, logarithmic solution is not allowed by regularity at $r = 0$ [13]) and $h(0) \neq 0$. The solution will still take the above form for $r > r_s$. Matching the solutions gives $B = A r_s^{2|nk|}$, and $C = 2 A r_s^{|nk|}$, thus $B$ can be neglected as before. Inserting this into (17) for $r_s < r < r_{ew}$ gives $b(r) = F r^{2(|nk|+1)} + G r^2$, where $G$ is arbitrary and $F$ is related to $A$. Matching $h$ and $b$ at $r = r_{ew}$ reveals that to first order $F$ can be neglected, and so a sensible trial solution is

$$
h(r) = \left\{ \begin{array}{c} \left(\frac{r}{r_{ew}}\right)^{|nk|} \\ 1 \end{array} \right. \qquad b(r) = \left\{ \begin{array}{cc} \left(\frac{r}{r_{ew}}\right)^2 & r < r_{ew} \\ 1 & r > r_{ew} \end{array} \right. \tag{18}
$$

For $m = 0$ this determines $C$ to be $2(r_s/r_{ew})^{|nk|}$, which although not equal to zero, is very close. An estimate of the energy of this string-like solution can now be found by substituting the trial solution into the Lagrangian. The contribution from the region $r < r_s$ is suppressed by powers of $r_s$ and can be neglected. All the contributions are zero for $r > r_{ew}$, thus

$$
\begin{aligned}
\text{Energy} &= 2\pi \int_{r_s}^{\infty} r \, dr \left\{ |\partial_r \Phi_{10}|^2 + |D_\theta \Phi_{10}|^2 + \tfrac{1}{2} (\partial_r A_\theta^a)^2 + V(\Phi_{10}) \right\} \\
&= 2\pi \int_{r_s}^{r_{ew}} r \, dr \left\{ \tfrac{\eta_2^2}{2} h'^2 + \tfrac{\eta_2^2 n^2 k^2}{2r^2} [1-b]^2 h^2 \right. \\
&\qquad\qquad\qquad \left. + \tfrac{5n^2 k^2}{4g^2} \left(\tfrac{b'}{r}\right)^2 + \lambda_2 \tfrac{\eta_2^4}{4} (h^2 - 1)^2 \right\} \\
&= 2\pi \eta_2^2 \left\{ \tfrac{|nk|}{4} + \tfrac{n^2 k^2}{2} \left( \tfrac{1}{2|nk|} - \tfrac{2}{2+2|nk|} + \tfrac{1}{4+2|nk|} \right) \right\} \\
&\quad + \tfrac{5n^2 k^2}{2g^2 r_{ew}^2} + \lambda_2 \tfrac{\eta_2^4 r_{ew}^2}{4} \left( \tfrac{1}{2} - \tfrac{2}{2|nk|+2} + \tfrac{1}{4|nk|+2} \right)
\end{aligned} \tag{19}
$$

Thus, taking $\eta_2 r_{ew} \sim 1$, the energy is of order $|nk|\eta_2^2$. It is minimised by minimising $|k|$, subject to the restriction that $m = n\left(k - \tfrac{1}{5}\right)$ is an integer. For $|n| < 3$ this occurs at $k = \tfrac{1}{5}$, giving $m = 0$. Thus for small $n$, although the electroweak Higgs field has a string-like profile (it is approximately zero at $r = 0$), it does not wind around the string. This is in agreement with earlier work by Alford and Wilczek [16]. The electroweak symmetry is restored around it, and over a much larger region than the other symmetry restoration, since $r_{ew} \gg r_s$. Putting $a(r) = 1$ in (16) and (17) will give the Nielsen-Olesen equations for a string with winding number $|nk|$, thus the profile of $h(r)$ is not what would usually be expected of an abelian string.

If $n$ is a multiple of 5 then the energy will be minimised by $k = 0$ (as would be expected). In this case $h$ is constant for $r > r_s$. Thus the approximate solution obtained with the above method is $h(r) = C r^{|n|/5}$ for $r < r_s$ and $h(r) = 1$ for $r > r_s$. This has approximately same form as $f(r)$. Although the electroweak symmetry is still restored, it is over a much smaller region than in the $k \neq 0$ cases. Throughout this region the electroweak Higgs field winds. All of these results are completely independent of the choice of $\kappa$ and $\tilde{\kappa}$.

9

Intuitively, it seems that the above solution has minimal energy. However, it might be possible to get even lower energy by adding more or different $SU(5)$ gauge fields. If (for $|n| < 3$) such gauge fields caused the Higgs field to wind (in order to give a zero covariant derivative at infinity), the energy would be higher, since it is roughly proportional to winding number. Thus an alternative gauge field ($\tilde{R}$) would have to satisfy $(T_s + \tilde{R})\mathcal{H}^0 = 0$ to give $e^{i\theta[T_s + \tilde{R}]}\mathcal{H}^0 = \mathcal{H}^0$ for all $\theta$. So $\tilde{R}\mathcal{H}^0 = -\frac{1}{5}\mathcal{H}^0$ is needed, which has the unique solution $\tilde{R} = R$, and so the minimal energy solution has been found.

## 4.2 The Nonabelian $SU(2)$ Strings

### 4.2.1 $X_S$ and $X'$ Strings

Although at the GUT scale all the $SU(2)$ strings have the same properties, this is not true at low temperatures, since they affect $\Phi_{10}$ differently. There are basically two different cases. The generators corresponding to the $X_{Si}^{\pm}$ and $X_i'^{\pm}$ both annihilate the usual vacuum expectation value of $\Phi_{10}$, and so the string gauge fields have no effect on this symmetry breaking, and $\Phi_{10}$ can be constant everywhere. There is still the possibility of symmetry restoration from potential terms, which is due to variation of the $\Phi_{126}$ and $\Phi_{45}$ fields (see section 5.3), although this is less significant. The other generators do have a non-trivial effect.

### 4.2.2 $W$ Strings

For the string with the $W_R^{\pm}$ gauge fields, $T_s^2\mathcal{H}^0 = \frac{1}{4}\mathcal{H}^0$, so $e^{2\pi i n T_s}\phi_{10}^{(0)} = (-1)^n\phi_{10}^{(0)}$. Thus, in the presence of a topological string (when $n$ is odd), as with the abelian string (when $\frac{n}{5}$ in not an integer), it is necessary to replace $T_s$ with $T_s + R$. $R$ is chosen to give a single valued $\Phi_{10}$, while still giving a vanishing covariant derivative at infinity. In order not to alter the GUT string $Ru$ and $RT_su$ must both be zero. The only possibility is $R = k_i\tau^{W_L^i}$, where $\tau^{W_L^i}$ is the generator corresponding to the $W_L^i$ boson. As with the GUT nonabelian string, the Higgs field needs to be split up into different eigenvectors of $n^2(R + T_s)^2$. Noting that $R^2\mathcal{H}^0 = k^2\mathcal{H}^0$ and $RT_s\mathcal{H}^0 = T_sR\mathcal{H}^0$, where $k^2 = \sum_i k_i^2$, it is found that for non-zero $k$, $\Phi_{10}$ is made up of two such eigenvectors

$$\psi_{p,q} = \frac{1}{2}\mathcal{H}^0 \pm \frac{1}{k}RT_s\mathcal{H}^0 \tag{20}$$

which have eigenvalues $p, q = n\left|\frac{1}{2} \pm k\right|$. Both of these must be integers, and so if $n$ is odd $2nk$ must be an odd integer to ensure that $\Phi_{10}$ is single valued.

If $n$ is even, it is sufficient for $nk$ to be an integer. In this case $k = 0$ is a possibility, and there is no need for extra gauge fields. The field equations are solved using the ansatz of the form $\Phi_{10} = \frac{\eta_2}{\sqrt{2}}e^{inT_s\theta}\mathcal{H}^0h(r)$.

10

Without loss of generality $k > 0$ and $p > q$ can be assumed when $n$ is odd. This gives the following ansatz for $\Phi_{10}$ and the modified gauge as

$$
\begin{aligned}
\Phi_{10} &= \tfrac{\eta_2}{\sqrt{2}} e^{in(T_s+R)\theta} \left( \psi_p h_p(r) + \psi_q h_q(r) \right) \\
A_\theta &= n\tfrac{2}{g} \left( \tfrac{a(r)}{r} T_s + \tfrac{b(r)}{r} R \right) \qquad\qquad A_\mu = 0 \;\; \text{otherwise}
\end{aligned} \tag{21}
$$

As with the other solutions, $h_p(\infty) = h_q(\infty) = b(\infty) = 1$, and $h_p(0) = b(0) = 0$ to give the correct asymptotic form of the solution and regularity at $r = 0$. If $q \neq 0$, $h_q(0) = 0$ is needed too. The resulting field equations are

$$
h_p'' + \frac{h_p'}{r} - n^2 \frac{\left[ \frac{1}{2}(1-a) + k(1-b) \right]^2}{r^2} h_p = \frac{2}{N_p \eta_2^2} \frac{\partial V}{\partial h_p} \tag{22}
$$

$$
h_q'' + \frac{h_q'}{r} - n^2 \frac{\left[ \frac{1}{2}(1-a) - k(1-b) \right]^2}{r^2} h_q = \frac{2}{N_q \eta_2^2} \frac{\partial V}{\partial h_q} \tag{23}
$$

$$
b'' - \frac{b'}{r} = -g^2 \eta_2^2 \left\{ \left[ (1-b) + \frac{1-a}{2k} \right] N_p h_p^2 + \left[ (1-b) - \frac{1-a}{2k} \right] N_q h_q^2 \right\} \tag{24}
$$

where $N_{p,q} = |\psi_{p,q}|^2$, and so $N_p + N_q = 1$. Except for the extra gauge term, these resemble the usual nonabelian vortex equations (10-12), like (16) and (17) resembled (13) and (14). As with the abelian string the energy can be estimated and minimised with a trial solution. When $r < r_{ew}$ the effect of $b$ on $h_p$, $h_q$ can be neglected, and $a$ can be taken to be 0 and 1 for $r$ less than and greater than $r_s$ respectively. Thus for $r < r_s$, $h_p = C_p r^p$ and $h_q = C_q r^q$. These contribute $r^{p+2}$ and $r^{2q+2}$ terms to the gauge term $b$, which also has an arbitrary $r^2$ term. For $r > r_s$, $h_{p,q} = A_{p,q} r^{nk} + B_{p,q} r^{-nk}$. Matching the solutions at $r = r_s$ gives $B_{p,q} \sim A_{p,q} r_s^{2nk}$, thus $B_{p,q}$ can be neglected. The gauge field then takes the form $b(r) = F r^{2nk+2} + G r^2$, with $G$ arbitrary and $F$ related to $B_p$ and $B_q$. Matching solutions at $r = r_{ew}$ shows that the $r^2$ term dominates. This suggests the trial solution

$$
h_p(r) = h_q(r) = \left\{ \begin{array}{c} \left( \frac{r}{r_{ew}} \right)^{nk} \\ 1 \end{array} \right. \quad b(r) = \left\{ \begin{array}{cc} \left( \frac{r}{r_{ew}} \right)^2 & r < r_{ew} \\ 1 & r > r_{ew} \end{array} \right. \tag{25}
$$

where, as in (18), the solution is assumed to take its asymptotic form for $r > r_{ew}$, and the arbitrary constants are determined by ensuring continuity. Substituting (21) into the Lagrangian gives an additional contribution to the energy (neglecting $r < r_s$) of

$$
\begin{aligned}
\text{Energy} = \quad & 2\pi \int_{r_s}^{r_{ew}} r\,dr \Big\{ \tfrac{\eta_2^2}{2} (N_p h_p'^2 + N_q h_q'^2) + \tfrac{\eta_2^2 n^2 k^2}{2r^2} [1-b]^2 (N_p h_p^2 + N_q h_q^2) \\
& + \tfrac{n^2 k^2}{2g^2} \left( \tfrac{b'}{r} \right)^2 + V(h_p, h_q) \Big\}
\end{aligned} \tag{26}
$$

11

However, for $r > r_s$ the trial solution (25) has $h_p(r) = h_q(r) = h(r)$ where $h(r)$ is the Higgs part of the abelian trial solution (18). Substituting this into (26) reduces it to an expression almost identical to (19), the extra abelian string energy. Thus, taking $\eta_2 r_{ew} \sim 1$, the electroweak $W$-string energy is of order $nk\eta_2^2$. For odd $n$, $2nk$ must be a positive odd integer, so the minimal energy occurs when $k = \frac{1}{2n}$, and so $p = \frac{1}{2}(n+1)$, $q = \frac{1}{2}(n-1)$. For even $n$ the energy is minimised by putting $k = 0$, and so there is no symmetry restoration outside of the GUT string radius ($r_s$). Thus only topological nonabelian $W$-strings will cause electroweak symmetry restoration outside of the string core. This is in contrast to the abelian case, in which symmetry is restored in the region $r > r_s$ for odd and even $n$.

The lowest energy string ($n = 1$) will have $k = \frac{1}{2}$, although this does not imply that the abelian string (with $|k| = \frac{1}{5}$) has lower energy than the nonabelian string, since there is also the contribution from the GUT part of the string, which is of order $\eta_1^2 \gg \eta_2^2$, and is higher for the abelian string. Despite the fact that only half the electroweak Higgs field winds arround the string, in a similar way to the GUT Higgs field, all of the Higgs field is approximately zero at the string's centre ($h_0 \sim \sqrt{r_s/r_{ew}}$ for $r < r_s$), and so the electroweak symmetry is almost fully restored there. In contrast, the symmetry breaking caused by the GUT Higgs field is only partially restored at the centre of the string, because $|\Phi_{126}|$ is about $\frac{1}{2}$ there. As with the abelian string the profile of $\Phi_{10}$ is like that of a string with a non-integer winding number, in this case $\frac{1}{2}$.

Unlike the abelian string there is a choice of extra gauge fields. For $r > r_s$ they all give approximately the same contribution to the energy, so it is necessary to consider the $r < r_s$ contribution to find the precise minimal energy solution. Taking $k = \frac{1}{2n}$ and matching solutions at $r = r_s$, gives $h_{p,q} = \sqrt{\frac{r_s}{r_{ew}}} \left(\frac{r}{r_s}\right)^{p,q}$. The extra energy contribution (ignoring potential and gauge terms) is then

$$2\pi \frac{\eta_2^2}{2} \frac{r_s}{r_{ew}} [N_p p + N_q q] \tag{27}$$

Thus, since $p > q$, the energy will be minimised by minimising $N_p$. $N_{p,q}$ can be found explicity in terms of the parameters of the gauge fields. $T_s$ is equal to $\frac{1}{2\sqrt{2}}(z\tau^{W_R^+} + z^*\tau^{W_R^-})$, for some $z$ with $|z| = 1$, and $R = k_i \tau^{W_L^i}$, with $\sqrt{\sum_i k_i^2} = k = \frac{1}{2n}$. Substituting these expressions into (20), and evaluating it gives

$$\psi_{p,q} = \frac{1}{2}[\kappa \pm \tilde{\kappa} z^*(k_1 + ik_2)/k]H^0 + \frac{1}{2}[\tilde{\kappa} \pm \kappa z(k_1 - ik_2)/k]\tilde{H}^0 \\ \pm \frac{1}{2}(z^*\tilde{\kappa}H^+ - z\kappa\tilde{H}^+)k_3/k \tag{28}$$

and so $N_{p,q} = \frac{1}{2} \pm \kappa\tilde{\kappa}\mathrm{Re}\{z(k_1 - ik_2)/k\}$. Working in a $U(1)_Q$ gauge in which $z$ is equal to 1, $N_p = \frac{1}{2} + \kappa\tilde{\kappa}k_1/k$. Since $|k_1| \leq k$, $N_p$ is minimised by putting $k_1 = -k = -\frac{1}{2n}$, which gives $k_2 = k_3 = 0$. Thus the minimal energy solution has only non-zero $W_L^\pm$ gauge fields. Furthermore the ratio $W_L^2/W_L^1$ is equal to $W_R^2/W_R^1$. The energy difference between the various choices of $k_i$ is only about

12

$\frac{r_s}{r_{ew}} \sim 10^{-13}$ of the electroweak contribution, and so it is quite possible other neglected effects could alter this choice.

The above is generally true, although if $\kappa$ or $\tilde{\kappa}$ is zero, the different choices of $R$ all give the same energy. They are all gauge equivalent under $SU(2)$, but since this symmetry is broken, there are several physically distinct minimal energy solutions.

An interesting case occurs when $\kappa = \tilde{\kappa}$. Then $\psi_p = 0$ and $\psi_q = \mathcal{H}^0$ giving

$$\Phi_{10} = \frac{\eta_2}{\sqrt{2}} h_q(r) e^{in(T_s + R)\theta} \mathcal{H}^0 \tag{29}$$

so $\Phi_{10}$ has just one winding number, instead of the usual two. In this respect it resembles the equivalent abelian case, and will share some of the properties of abelian strings (see section 6). There is a slight difference in that $(T_s + R)\mathcal{H}^0$ is not proportional to $\mathcal{H}^0$.

### 4.2.3 $Y$ Strings

The fourth type of nonabelian string has $T_s$ proportional to an appropriate linear combination of the generators of the $Y_i'^{\pm}$ gauge fields. In this case $T_s^2 \mathcal{H}^0 = \frac{1}{4}\mathcal{H}^0$, and so the situation is similar to the $W$-string, and has a similar solution. $T_s$ will be equal to $\frac{1}{2\sqrt{2}} \left( c_i \tau^{Y_i'^{+}} + c_i^* \tau^{Y_i'^{-}} \right)$, with $\sum_i |c_i|^2 = 1$. To give a single valued $\Phi_{10}$, $T_s$ must be replaced with $T_s + R$, where $R$ does not alter the GUT string, but still affects $\Phi_{10}$. The only such generators are $c_i \tau^{Y_i^{+}}$, $c_i^* \tau^{Y_i^{-}}$, and a third generator made up of the $A$, $Z$, and gluon fields. Together they form an $SU(2)$ subgroup, with the third generator the equivalent of $\tau^{W_L^3}$. Like the $W$-string, the different choices of $R$ give an approximately equal increase in energy at the electroweak scale. However, $\tau^{Y_i^{\pm}}$ have a non-trivial effect on $\Phi_{45}$, so when the $SU(5)$ symmetry breaking is taken into account, the choice of $R$ that minimises the energy will be different to the corresponding choice for the $W$-string (see section 5.1). The $Y$-string is less physically significant than the $W$-string, since although it gives a globally defined charge, it gives multi-valued $W_L^{\pm}$ fields.

## 4.3 Summary

Electroweak symmetry is restored and electroweak string gauge fields are present in the presence of the abelian and the $W$ and $Y$ nonabelian GUT strings. This generally occurs in a region around the string whose size is inversely proportional to the electroweak Higgs VEV, and is much bigger than the string core. If $n$ is a multiple of 5 for the abelian, or 2 for the nonabelian strings, the region is approximately the same as the string core, and there are no extra string gauge fields.

It is also possible that $\Phi_{10}$ will wind. For the abelian string its winding number is the closest integer to $-\frac{1}{5}n$, and hence zero for $n = 1$. For the $W$ and

13

$Y$ nonabelian strings it is $\frac{1}{2}n$ for even $n$. For odd $n$, different parts of $\Phi_{10}$ have different winding numbers, a bit like the corresponding GUT string. They are $\frac{1}{2}(n-1)$ and $\frac{1}{2}(n+1)$. The remaining two nonabelian strings ($X_S$ and $X'$) have no effect on $\Phi_{10}$ at all.

# 5    Other Symmetry Restorations

## 5.1    The Intermediate Symmetry Restoration

So far the effect of the string on the second Higgs field $\Phi_{45}$ has been neglected, because it is far less significant. $\Phi_{45}$ is in the adjoint representation, and so its covariant derivative takes the form $D_\mu \Phi_{45} = \partial_\mu \Phi_{45} - \frac{1}{2} ig\, [A_\mu, \Phi_{45}]$. The generator corresponding to the $S$ particle ($P$) commutes with $\Phi_{45}$, so the gauge fields of the abelian string will not stop $\Phi_{45}$ from taking its usual vacuum expectation value everywhere. Thus it has no effect on the electroweak symmetry breaking, and gives no additional contribution to the energy. The other strings will give non-vanishing covariant derivatives at infinity. This is avoided by allowing $\Phi_{45}$ to wind like a string

$$\Phi_{45} = e^{in\theta T_s}\phi_{45}^{(0)}(r)e^{-in\theta T_s} \tag{30}$$

where $\phi_{45}^{(0)}(\infty)$ is equal to the usual vacuum expectation value of $\Phi_{45}$. Conveniently $e^{2\pi n i T_s}$ (for all choices of $T_s$) and $\phi_{45}^{(0)}$ commute, so $\Phi_{45}$ will be single valued for all $n$, and no extra gauge terms are needed. As with the $\Phi_{126}$ and $\Phi_{10}$ fields it is necessary to split $\Phi_{45}$ up into eigenstates of $T_s^2$. Thus, using the fact that $\phi_{45}^{(0)}(\infty)$ and $T_s^2$ commute, and that $T_s^3 = \frac{1}{4}T_s$

$$\Phi_{45} = e^{in\theta T_s}\psi_1 s_1(r)e^{-in\theta T_s} + \psi_0 s_0(r)$$
$$\text{with} \qquad \psi_1 = 2T_s^2\phi_{45}^{(0)}(\infty) - 2T_s\phi_{45}^{(0)}(\infty)T_s, \qquad \psi_0 = \phi_{45}^{(0)}(\infty) - \psi_1 \tag{31}$$

The field equations for $s_{0,1}(r)$ will be similar to (10) and (11), with the similar boundary conditions $s_{0,1}(\infty) = 1$ and $s_1(0) = 0$. Since the gauge contribution $(1 - a(r))$ vanishes for $r > r_s$, $\phi_{45}^{(0)}(r) = \phi_{45}^{(0)}(\infty)$ is a solution outside the string. Thus the region of symmetry restoration will be of radius $r_s$ (order $|\Phi_{126}(\infty)|^{-1}$). This is in contrast to the other Higgs fields, which restore symmetry in regions of order the reciprocal of their own values at $r = \infty$.

This is true for the $X'$, $X_S$ and $W$-strings. For higher temperatures, at which the $\Phi_{10}$ symmetry breaking has not occured, it is also true for the $Y$-string. After the final symmetry breaking, there is an additional contribution from the $Y_i^{\pm}$ fields. This can be resolved by altering (30) to $\Phi_{45} = e^{in\theta(T_s+R)}\phi_{45}^{(0)}(r)e^{-in\theta(T_s+R)}$, which is still single valued. If $R\phi_{45}^{(0)}(\infty) \neq 0$ this will give a large increase in the energy. $R\phi_{45}^{(0)}(\infty)$ is only equal to zero when $R$ involves just the $Y$-string equivalent of $\tau^{W_L^3}$, so this possibility is most likely to occur. With the $W$-string, $R\phi_{45}^{(0)} = 0$ for all $R$, since $W_L^{\pm}$ and $\Phi_{45}$ commute,

so $\Phi_{45}$ does not alter the energetically prefered choice of $R$. Thus as previously stated, the $\Phi_{45}$ symmetry breaking has little effect on the string properties, except for the $Y$-string.

## 5.2   The Minimal Energy Choice of $T_s$

After the first symmetry breaking, there were only two gauge inequivalent possible strings, and the nonabelian strings had the lowest energy. After the second symmetry breaking, the $X'$ and $Y$-strings, which are gauge equivalent under $SU(2)_L$ have the lowest energy. Of the nonabelian strings, the $W$-string has highest energy, with the $X_S$-string having slightly less. The final symmetry breaking gives an additional contribution to the $Y$-string, so the most energetically favourable choice of string generator is made up of $\tau^{X_i'^{\pm}}$, and does not have any effect on electroweak symmetry. Just because the other strings are not energetically favourable does not mean that they will not form, but just that they are less likely to form (but see section 7).

## 5.3   Non-gauge field symmetry restoration

Even when a Higgs field is unaffected by a string's gauge fields, it is still possible for symmetry restoration to occur via the potential terms. This has previously been discussed for an abelian string in [5]. For example, in the absence of strings, before electroweak symmetry restoration occurs, so $\Phi_{10} = 0$, the potential takes the form

$$V(\Phi_{126}, \Phi_{45}) = \lambda_1 \left( |\Phi_{126}|^2 - \frac{\eta_1^2}{2} \right)^2 + \lambda' \left( |\Phi_{45}|^2 - \frac{\eta'^2}{2} \left[ \alpha + \beta \frac{2}{\eta_1^2} |\Phi_{126}|^2 \right] \right)^2 \tag{32}$$

where $\eta'/\sqrt{2}$ is the usual VEV of $\Phi_{45}$, and $\alpha + \beta = 1$. This is minimised by setting $\Phi_{126}$ and $\Phi_{45}$ to their usual VEVs. However, in the presence of an abelian string $|\Phi_{126}|$ is proportional to $f(r)$, so, writing $|\Phi_{45}| = s(r)\eta'/\sqrt{2}$, the potential becomes

$$\lambda_1 \frac{\eta_1^4}{4} (f^2 - 1)^2 + \lambda' \frac{\eta'^4}{4} (s^2 - [\alpha + \beta f^2])^2 \tag{33}$$

which, for small $r$, is no longer minimised by $s = 1$, for $\beta \neq 0$. Thus, if the theory's parameters take appropriate values, $|\Phi_{45}|$ will be lower than usual, or even zero at the string's centre. Alternatively $|\Phi_{45}|$ could be higher than usual there.

If symmetry restoration by this mechanism occurs at all, it will only be in the region $r < r_s$, since $|\Phi_{126}|$, and hence $V(\Phi_{126}, \Phi_{45})$ take their usual values at higher $r$. Unlike the corresponding symmetry restoration by gauge fields, $\Phi_{45}$ will never wind. When $\Phi_{10}$ is not equal to its usual VEV, it could also cause

15

$|\Phi_{45}|$ to vary. In this case the symmetry restoration could take place in the larger $r < r_{ew}$ region.

A similar situation can occur with $\Phi_{10}$ in the presence of an $X'$ or $X_S$ string. In this case both $|\Phi_{45}|$ and $|\Phi_{126}|$ are lower than usual for $r < r_s$. For nonabelian strings the potential is more complicated since it involves $f_1$ and $f_0$ terms, as well as the corresponding $\Phi_{45}$ terms. Even inside the string, $f_0$ and $s_0$ are non-zero, so the variation of the potential is likely to be less substantial than the abelian case, and hence extra symmetry restoration is less likely to occur. Even if it does, $\Phi_{10}$ will always take its usual VEV outside the string.

## 6    Fermion Zero Modes

In the presence of a string, non-trivial zero energy solutions to the fermion equations of motion may exist. If such solutions involve charged fermions moving along the string, they will correspond to a superconducting current. The effects of such a current will be observable at large distances, thus providing a possible method of observing cosmic strings [11]. The fermionic part of the Lagrangian is

$$\mathcal{L}_{\text{fermions}} = \bar{\Psi}_L i\gamma^\mu D_\mu \Psi_L - \frac{1}{2}ig_2\bar{\Psi}_L\Phi_{10}\Psi_L^c - \frac{1}{2}ig_1\bar{\Psi}_L\Phi_{126}\Psi_L^c + (\text{h. c.}) \quad (34)$$

where $\gamma^\mu = (1, \sigma^i)$ since $\Psi_L$ is a two component spinor. Varying $\bar{\Psi}_L$ in (34) gives

$$i\gamma^\mu D_\mu \Psi_L - ig_2\Phi_{10}i\sigma_2\Psi_L^* - ig_1\Phi_{126}i\sigma_2\Psi_L^* = 0 \quad (35)$$

Because of the symmetries of the system, it is possible to look for solutions which are independent of $z$ and $t$. It is also possible to separate out their $r$ and $\theta$ dependence. If such solutions are found, it is easy to generalise them and allow $z$ and $t$ dependence. The resulting equations are easily solved, and correspond to superconducting currents [11]. Thus the existence of two-dimensional solutions implies that the string is superconducting. Following the approach of Jackiw and Rossi [9], $\Psi_L$ can be split into eigenvectors of $\sigma_3$, $\psi_U$ and $\psi_D$, and (35) becomes

$$e^{i\theta}\left(\partial_r + \frac{i}{r}\partial_\theta + \frac{g}{2}A_\theta\right)\psi_U + g_2\Phi_{10}\psi_U^* + g_1\Phi_{126}\psi_U^* = 0 \quad (36)$$

$$e^{-i\theta}\left(\partial_r - \frac{i}{r}\partial_\theta - \frac{g}{2}A_\theta\right)\psi_D - g_2\Phi_{10}\psi_D^* - g_1\Phi_{126}\psi_D^* = 0 \quad (37)$$

Jackiw and Rossi considered the affect of an abelian string on a system in which only one Higgs field was present. The presence of additional Higgs fields complicates things. However $\Phi_{126}$ has no effect on most of the fermion fields, so the results of [9] can be applied in some cases.

It is not necessary to attempt to solve both (37) and (36), since one can be transformed into the other. The complex conjugate of (36), with the substitutions $n \rightarrow -n$ and $\psi_U \rightarrow i\psi_D^*$, gives (37). Thus, although nonabelian strings with winding numbers $n$ and $-n$ are gauge equivalent, it is convenient to consider them both when solving (36).

## 6.1 Zero Modes for the Abelian String

### 6.1.1 High Temperature Neutrino Zero Modes

At high temperatures $\Phi_{10}$ is zero, and so with the exception of $\nu^c$, none of the fermion fields are affected by Higgs fields. For the conjugate neutrino (36) becomes

$$e^{i\theta}\left(\partial_r + \frac{i}{r}\partial_\theta + \frac{1}{2}n\frac{a(r)}{r}\right)\nu_U^c + m_1 f(r)e^{in\theta}\nu_U^{c*} = 0 \qquad (38)$$

where $m_1 = g_1\eta_1/\sqrt{2}$. This equation has been discussed in detail in [9]. It has $n$ normalisable solutions for $n > 0$, and none otherwise. A similar equation is obtained from (37) (or by transforming (38)). This has suitable solutions if $n < 0$, in which case there are a total of $|n|$ of them. Thus conjugate neutrino zero modes always exist at high temperatures in the presence of an abelian string, and there are $|n|$ of them. For $r > r_s$ the solutions decrease exponentially, so the zero modes are confined to the string core.

### 6.1.2 High Temperature Non-Neutrino Zero Modes

Although there is no Higgs field acting on the other fermion fields, it is possible for zero modes to be generated by the string gauge fields, as discussed by Stern and Yajnik [10]. (36) and (37) reduce to

$$\left(\partial_r + \sigma_{U,D}\left[\frac{i}{r}\partial_\theta + p_\lambda n\frac{a(r)}{r}\right]\right)\lambda_{U,D} = 0 \qquad (39)$$

where $\lambda = u_i, d_i^c$, etc. (not $\nu^c$), and $\sigma_{U,D} = \pm 1$. $p_\lambda$ is the eigenvalue of the field with respect to $\frac{P}{10}$, (so $p_{e^-} = -\frac{3}{10}$, $p_{u_i^c} = \frac{1}{10}$, etc.). There are normalisable solutions if $|np_\lambda| > 1$, all of which can be found analytically. The number of solutions is equal to the highest integer that is less than $|np_\lambda|$. Thus $|n|$ must be at least 4 for any zero modes of this type to exist. If the only stable strings have winding number 1, then only conjugate neutrino zero modes will be present at high temperatures around an abelian string.

### 6.1.3 Low Temperature Non-Neutrino Zero Modes

At lower temperatures $\Phi_{10}$ is non-zero and so the situation is different. With the exception of the neutrino fields, there is just one Higgs field coupling to

the fermions. Its two components have winding numbers $m$ and $-m$, and (36) reduces to

$$e^{i\theta}\left(\partial_r + \frac{i}{r}\partial_\theta + \frac{g}{2}A_\theta\right)\lambda_U^c + m_\lambda h(r)e^{\pm im\theta}\lambda_U^* = 0 \tag{40}$$

$$e^{i\theta}\left(\partial_r + \frac{i}{r}\partial_\theta + \frac{g}{2}A_\theta\right)\lambda_U + m_\lambda h(r)e^{\pm im\theta}\lambda_U^{c*} = 0 \tag{41}$$

where $\lambda$ can be $d_i$, $u_i$, or $e^-$, and $m_u = \tilde{\kappa}g_2\frac{\eta_2}{4\sqrt{2}}$, $m_e = m_d = \kappa g_2\frac{\eta_2}{4\sqrt{2}}$. The upper sign applies for the $d_i$, $d_i^c$ and $e^\pm$ fields, since they couple to the $H^0$ component of the Higgs field. The lower sign is taken for the $u_i$ and $u_i^c$ fields, which couple to the $\tilde{H}^0$ component. These equations are similar to those in [9], and have been discussed in [10]. For the down quark and electron fields, they have $m$ normalisable solutions per particle type, if $m > 0$ (which only occurs if $n < 0$). The corresponding equations for $\lambda_D$ have $-m$ solutions for $m < 0$. For the up quark fields, there are $-m$ (or $m$ for $u_{iD}$ equations) normalisable solutions per particle type for $u_{iU}$ ($u_{iD}$) if $m < 0$ ($m > 0$). If $n$ is not a multiple of 5, the solutions decay exponentially outside $r = r_{ew}$. When $n$ is a multiple of 5 (in which case $m = -\frac{1}{5}n$), they decay outside $r = r_s$. Thus the zero modes are confined to the region of symmetry restoration.

The difference in sign between up and down quarks in (40) and (41) has physical significance when time and $z$ dependence are added to the solutions. They then represent currents flowing along the string, with the up quark current flowing in the opposite direction to the down quark and electron currents.

### 6.1.4   Low Temperature Neutrino Zero Modes

The situation is more complex for the neutrino fields since they are affected by two Higgs fields at the same time. In this case (36) becomes

$$e^{i\theta}\left(\partial_r + \frac{i}{r}\partial_\theta + \frac{n}{2}\frac{a(r)}{r}\right)\nu_U^c + m_\nu h(r)e^{-im\theta}\nu_U^* + m_1 f(r)e^{in\theta}\nu_U^{c*} = 0 \tag{42}$$

$$e^{i\theta}\left(\partial_r + \frac{i}{r}\partial_\theta - \frac{3n}{10}\frac{a(r)}{r} - \left[m + \frac{n}{5}\right]\frac{b(r)}{r}\right)\nu_U + m_\nu h(r)e^{-im\theta}\nu_U^{c*} = 0 \tag{43}$$

where $m_\nu = m_u$. Although Jackiw and Rossi did not consider this case, it can be approached using a similar method to theirs. The angular dependence can be removed with the substitutions

$$\nu_U^c = Ae^{il\theta} + B^*e^{i(n-1-l)\theta} \tag{44}$$

$$\nu_U = C^*e^{-i(m+1+l)\theta} + De^{-i(m+n-l)\theta} \tag{45}$$

(The case $2l = n - 1$ will be considered later.) Ignoring the gauge terms, the resulting four complex differential equations are

$$\left(\partial_r - \frac{l}{r}\right)A + m_\nu h(r)C + m_1 f(r)B = 0 \tag{46}$$

18

$$\left(\partial_r - \frac{n-1-l}{r}\right)B + m_\nu h(r)D + m_1 f(r)A = 0 \tag{47}$$

$$\left(\partial_r + \frac{m+1+l}{r}\right)C + m_\nu h(r)A = 0 \tag{48}$$

$$\left(\partial_r + \frac{m+n-l}{r}\right)D + m_\nu h(r)B = 0 \tag{49}$$

Splitting these equations into real and imaginary parts gives two identical sets of four real equations, and so it is only necessary to look for real solutions. When $r$ is large, $h(r)$ and $f(r)$ are both approximately 1 and the gauge terms can be neglected as they are of order $\frac{1}{r}$. Eliminating $C$ and $D$, and then solving gives

$$A + B \propto e^{\frac{1}{2}\left(-m_1 \pm \sqrt{m_1^2 + 4m_\nu^2}\right)} \quad \text{and} \quad A - B \propto e^{\frac{1}{2}\left(m_1 \pm \sqrt{m_1^2 + 4m_\nu^2}\right)} \tag{50}$$

$C$ and $D$ are then proportional to $A$ and $B$ respectively. There are four linearly independent solutions to these equations. They all have exponential behaviour at $r = \infty$. Only two linear combinations are normalisable there.

Near the origin, to first order in $r$, $h(r) = Gr^{|m|}$, $f(r) = Fr^{|n|}$, and the gauge terms can be dropped. Making these substitutions, the four real (or imaginary) solutions of (46-49) near $r = 0$ can be found to first order

$$\begin{aligned}
A &\sim r^l, \quad r^{n+|n|-l}, \quad r^{|m|-m-l}, \quad r^{|m|-m+|n|-n+2+l} \\
B &\sim r^{1+|n|+l}, \quad r^{n-1-l}, \quad r^{|m|-m+|n|+1-l}, \quad r^{|m|-m-n+1+l} \\
C &\sim r^{|m|+1+l}, \quad r^{|m|+n+|n|+1-l}, \quad r^{-m-1-l}, \quad r^{2|m|-m+|n|-n+3+l} \\
D &\sim r^{|n|+|m|+2+l}, \quad r^{|m|+n-l}, \quad r^{2|m|-m+|n|+2-l}, \quad r^{-m-n+l}
\end{aligned} \tag{51}$$

If $\varphi$ is solution of (46-49) for all $r$, which is normalisable at $r = \infty$, then it will have to match some combination of the 2 normalisable solutions in (50) for large $r$. At $r = 0$, $\varphi$ will be made up of a combination of the solutions in (51). So if $\varphi$ is to be normalisable everywhere, at least 3 of the solutions (51) must be well behaved at $r = 0$. Thus for each $l$ satisfying 3 of the inequalities $l \geq 0$, $l \leq -m - 1$, $l \leq n - 1$ and $l \geq n + m$, there will be one normalisable solution. If $l$ satisfies all 4 there will be 2 solutions. Not all of these solutions are independent since the real (or imaginary) solutions for $l = l'$ and $l = n - 1 - l'$ are proportional.

For $l = \frac{1}{2}(n-1)$ the angular dependence of (42) and (43) is removed with the substitutions

$$\nu_U^c = Ae^{il\theta} \qquad\qquad \nu_U = C^* e^{-i(m+1+l)\theta} \tag{52}$$

giving (after dropping gauge terms) the equations

$$\left(\partial_r - \frac{l}{r}\right)A + m_\nu h(r)C + m_1 f(r)A^* = 0 \tag{53}$$

$$\left(\partial_r + \frac{m+1+l}{r}\right) C + m_\nu h(r) A = 0 \tag{54}$$

These have $A$ and $C$ proportional to $e^{\left(m_1\sigma \pm \sqrt{m_1^2 + 4m_\nu^2}\right)}$ for large $r$, with $\sigma = \pm 1$ depending on whether the real or imaginary parts of $A$ and $C$ are being considered. For small $r$

$$\begin{aligned} A &\sim r^l, \quad r^{|m|-m-l} \\ C &\sim r^{|m|+1+l}, \quad r^{-m-1-l} \end{aligned} \tag{55}$$

So in this case there is one real and one imaginary solution if $0 \le l \le -m-1$.

This gives a grand total of $-2m$ ($-m$ real and $-m$ imaginary) normalisable solutions if $m < 0$, and 0 otherwise. Surprisingly this does not depend on $n$. A similar approach can be applied to (37) to give $2m$ normalisable solutions, provided $m > 0$. Hence there are $2|m|$ possible neutrino zero modes after electroweak symmetry breaking. As with the other particle zero modes, they will be confined to the region of symmetry restoration.

For a topologically stable string $m = 0$, so no neutrino zero modes form. This is slightly surprising, since at higher temperatures, when $\Phi_{10}$ is zero, the abelian string does have neutrino zero modes, and intuitively, since $\Phi_{10} \sim \frac{\eta_2}{\eta_1}\Phi_{126} \sim 10^{-13}\Phi_{126}$, the situation would be the same for lower temperatures.

## 6.2 Zero Modes of the Nonabelian Strings

There are two additional complications with nonabelian strings. Firstly the particle states are not eigenstates of the string generator, although this is easily solved by re-expressing the problem in terms of gauge eigenstates. Secondly, there are effectively twice as many Higgs fields, since each Higgs field has two parts with different winding numbers and different profiles.

### 6.2.1 High Temperature Neutrino Zero Modes

At high temperatures the gauge fields are proportional to $T_s$. Since $\nu^c$ is not an eigenstate of $T_s$, the equivalent of (38) is obtained by putting $\Psi_L = (\hat{\nu}^c \pm 2T_s\hat{\nu}^c)\chi^{(\pm)}(r,\theta)$, which are eigenvectors of $T_s$. Their eigenvalues are $\pm\frac{1}{2}$. Substituting this and (9) into (36) gives

$$e^{i\theta}\left(\partial_r + \frac{i}{r}\partial_\theta \pm \frac{na(r)}{2r}\right)\chi_U^{(\pm)} + m_1\frac{1}{2}\left(f_0(r)\chi_U^{(\mp)*} + e^{\pm in\theta}f_1(r)\chi_U^{(\pm)*}\right) = 0 \tag{56}$$

(The equivalent equations from (37) can be obtained by complex conjugation.) These two equations bear some resemblance to (42) and (43), and the $\theta$ dependence can be removed with the substitutions

$$\chi_U^{(+)} = Ae^{il\theta} + B^*e^{i(n-1-l)\theta} \tag{57}$$

$$\chi_U^{(-)} = C^* e^{-i(l+1)\theta} + D e^{i(l-n)\theta} \tag{58}$$

which are the same as (44) and (45), when $m$ is equal to zero. The resulting $r$ dependent equations closely resemble (46-49). Similarly they also have four independent solutions, only two of which are well behaved at infinity, the rest increase exponentially. For small $r$ they behave as badly as (51) with $m = 0$. Thus there are no normalisable solutions for any values of $n$.

### 6.2.2 High Temperature Non-Neutrino Zero Modes

For the fermion fields that do not couple to $\Phi_{126}$ it is possible for zero modes to exist by the same mechanism as (39). However, unlike the abelian case, some fermion fields are annihilated by $T_s$ so $p_\lambda$ is effectively zero, and they cannot have zero energy solutions for any value of $n$. For instance, the $u_i$, $d_i$, $\nu$ and $e^-$ fields are all zero eigenvectors of the string generator for the high temperature $W$-string. Thus solutions can only occur for the conjugate fields, in the presence of this type of string. Putting $\Psi_L = (\hat{\lambda} \pm 2T_s\hat{\lambda})\chi^{(\pm)}(r,\theta)$, where $T_s\hat{\lambda}$ is not proportional to $\hat{\nu}^c$ or $T_s\hat{\nu}^c$, or equal to zero, the nonabelian equivalent of (39) is

$$\left(\partial_r + \sigma_{U,D}\left[\frac{i}{r}\partial_\theta \pm \frac{1}{2}n\frac{a(r)}{r}\right]\right)\chi_{U,D}^{(\pm)} = 0 \tag{59}$$

It has the solutions

$$\chi_{U,D}^{(\pm)} = r^l \exp\left(\sigma_{U,D}\left\{il\theta \mp \frac{n}{2}\int_0^r ds\frac{a(s)}{s}\right\}\right) \tag{60}$$

The solutions are normalisable if $0 \leq l < \pm\sigma_{U,D}\frac{n}{2} - 1$. Thus the total number of solutions, per number of particles (6 in this case), will be the largest integer below $\frac{1}{2}n$. In order for any such solutions to exist $n$ must be at least 3, so they do not occur for topologically stable strings.

### 6.2.3 Low Temperature $X_S$ and $X'$ String Zero Modes

At low temperatures $\Phi_{10}$ is non-zero and couples to all the fermion fields. When $T_s$ is made up of generators of the $X_S$ or $X'$ fields, $\Phi_{10}$ just takes its usual vacuum expectation value. For the fermion fields that are not affected by $\Phi_{126}$ there is effectively no string and so no zero modes. For the fields affected by $\Phi_{126}$ the solutions of the field equations will be at least as divergent as those of (56), so there will no normalisable solutions.

### 6.2.4 Low Temperature $W$-String Non-Neutrino Zero Modes

The neutrino and electron fields all couple to $\Phi_{126}$ in the presence of a $W$-string, while the quark fields are only affected by $\Phi_{10}$. For a topological $W$-string (so

$n$ is odd), $\Phi_{10}$ can be determined using (28) and the comments after it

$$
\begin{aligned}
\Phi_{10} &= \tfrac{\eta_2}{\sqrt{2}} e^{in\theta(T_s+R)} \left\{ \tfrac{\kappa-\tilde{\kappa}}{2}(H^0 - \tilde{H}^0)h_p + \tfrac{\kappa+\tilde{\kappa}}{2}(H^0 + \tilde{H}^0)h_q \right\} \\
&= \tfrac{\eta_2}{\sqrt{2}} \Big[ \tfrac{\kappa-\tilde{\kappa}}{2} \left\{ (H^0 - \tilde{H}^0)\cos p\theta + i(\tilde{H}^+ - H^+)\sin p\theta \right\} h_p \\
&\quad + \tfrac{\kappa+\tilde{\kappa}}{2} \left\{ (H^0 + \tilde{H}^0)\cos q\theta + i(\tilde{H}^+ + H^+)\sin q\theta \right\} h_q \Big]
\end{aligned}
\tag{61}
$$

$z$ has been gauge transformed to 1. Putting $\Psi_L = a_i \sum_{\pm} (\hat{u}_i \pm \hat{d}_i)\chi^{(\pm)} + (\hat{u}_i^c \pm \hat{d}_i^c)\chi^{c(\pm)}$ (where $a_i$ are real) and the expression for the Higgs field into (36), and using the fact that $\hat{u}_i \pm \hat{d}_i$ and $\hat{u}_i^c \pm \hat{d}_i^c$ are eigenvectors of $R$ and $T_s$, gives

$$
\begin{aligned}
e^{i\theta} \left( \partial_r + \tfrac{i}{r}\partial_\theta \pm \tfrac{na(r)}{2r} \right) \chi_U^{c(\pm)} & \\
+ \tfrac{\kappa+\tilde{\kappa}}{2} m_2 h_q e^{\pm iq\theta} \chi_U^{(\pm)*} &- \tfrac{\kappa-\tilde{\kappa}}{2} m_2 h_p e^{\pm ip\theta} \chi_U^{(\mp)*} = 0
\end{aligned}
\tag{62}
$$

$$
\begin{aligned}
e^{i\theta} \left( \partial_r + \tfrac{i}{r}\partial_\theta \mp \tfrac{b(r)}{2r} \right) \chi_U^{(\pm)} & \\
+ \tfrac{\kappa+\tilde{\kappa}}{2} m_2 h_q e^{\pm iq\theta} \chi_U^{c(\pm)*} &- \tfrac{\kappa-\tilde{\kappa}}{2} m_2 h_p e^{\mp ip\theta} \chi_U^{c(\mp)*} = 0
\end{aligned}
\tag{63}
$$

where $m_2 = g_2 \frac{\eta_2}{4\sqrt{2}}$. If $\kappa \neq \tilde{\kappa}$ the angular dependence can then be separated with the surprisingly simple substitutions

$$
\begin{array}{ll}
\chi_U^{(+)} = A e^{il\theta} & \chi_U^{(-)} = B e^{i(p-q+l)\theta} \\
\chi_U^{c(+)} = C^* e^{-i(1-q+l)\theta} & \chi_U^{c(-)} = D^* e^{-i(1+p+l)\theta}
\end{array}
\tag{64}
$$

where $l$ must be an integer. The resulting equations for $A, B, C, D$ have two sets of four solutions. One set is real, the other purely imaginary. They both satisfy the same real equations. As with (42) and (43) their behaviour for large and small $r$ can be found. The solutions have exponential behaviour at large $r$, two are divergent, two are normalisable. To first order for small $r$, four of the terms present (one from each solution), are proportional to $r^l$, $r^{p-q+l} = r^{1+l}$, $r^{q-1-l} = r^{\frac{1}{2}(n-3)-l}$ and $r^{-p-1-l} = r^{-\frac{1}{2}(n+3)-l}$. In order to match up with some combination of the normalisable large $r$ solutions, at least 3 of these must be well behaved at $r = 0$. This occurs when $q - 1 \geq l \geq 0$, in which case just 3 are well behaved. Thus there are $2q = n - 1$ (real and imaginary) normalisable solutions to (36), and (by complex conjugation) $2q$ solutions of (37).

If $\kappa = \tilde{\kappa}$, the $e^{\pm ip\theta}$ terms in (62) and (63) are not present. Apart from the gauge terms, these are practically the same as (40) and (41). They can be solved in the same way, so there are $2q$ normalisable solutions. The corresponding equations from (37) also have $2q$ solutions.

Since there are 3 linearly independent choices of $a_i(\hat{u}_i \pm \hat{d}_i)$ there are a total of $12q$ different zero modes for the $W$-string after electroweak symmetry breaking. The solutions are contained in the $r < r_{ew}$ region. Since $q = 0$ for the energetically stable $n = 1$ string, it has no fermion zero modes.

When $n$ is even (so the string is actually topologically equivalent to the vacuum), $\Phi_{10}$ is equal to

$$\frac{\eta_2}{\sqrt{2}}\left\{(H^0\kappa + \tilde{H}^0\tilde{\kappa})\cos\frac{n}{2}\theta + i(\tilde{H}^+\kappa + H^+\tilde{\kappa})\sin\frac{n}{2}\theta\right\}h_{n/2} \qquad (65)$$

This is the same as (61) if $p$ and $q$ are both set equal to $n/2$. Thus the results for the odd $n$ strings can be applied to even $n$ strings, and there are $12\frac{n}{2}$ normalisable solutions.

The mass terms considered in this theory are not of the same form as the ones usually used in $SU(5)$ GUTs. Usually the VEV of $\Phi_{10}$ consists of only one component in the $\mathbf{5}_{-2}$ representation (equivalent to putting $\kappa = 1$ and $\tilde{\kappa} = 0$). This gives masses to the down quarks and electrons. The up quarks (and in $SO(10)$, the neutrinos) get masses from the $P$-charge conjugate of $\Phi_{10}$, which transforms under the $\bar{\mathbf{5}}_2$ representation. The fermionic part of the Lagrangian is thus taken as

$$\mathcal{L}_{\text{fermions}} = \bar{\Psi}_L i\gamma^\mu D_\mu \Psi_L - \tfrac{1}{2}i\bar{\Psi}_L\left(g_2'\Phi_{10} + \tilde{g}_2'\tilde{\Phi}_{10}\right)\Psi_L^c \\ - \tfrac{1}{2}ig_1\bar{\Psi}_L\Phi_{126}\Psi_L^c + (\text{h. c.}) \qquad (66)$$

where $\tilde{\Phi}_{10}$ is the $P$-charge conjugate of $\Phi_{10}$. If $\Phi_{10} = \phi_\alpha H^\alpha$, then $\tilde{\Phi}_{10}$ is equal to $\phi_\alpha^* \tilde{H}^\alpha$. If $g_2'$ and $\tilde{g}_2'$ are suitably defined, (66) will be equal to (34) in the vacuum, or in the presence of an abelian string, so the two theories will be equivalent.

If a nonabelian string which affects $\Phi_{10}$ is present, the two theories are different. If it is a $W$-string, $R$ will be equal to $k_i\tau^{W_L^i}$. Since $\tilde{\kappa} = 0$, the only restriction on $k_i$ is $k = \frac{1}{2n}$. Choosing a gauge in which $T_s = \frac{1}{2\sqrt{2}}[\tau^{W_R^+} + \tau^{W_R^-}]$, and taking $n = 1$, $g_2' = \tilde{g}_2'$, $k_1 = 1$ and $k_2 = k_3 = 0$, gives (from (28))

$$\Phi_{10} + \tilde{\Phi}_{10} = \frac{\eta_2}{\sqrt{2}}(H^0 + \tilde{H}^0)h_1\cos\theta \qquad (67)$$

which is proportional to the quark and electron mass terms. This implies the fermion masses vary with $\theta$, and actually vanish at $\theta = \pm\frac{\pi}{2}$. Moreover this is true for all $r$. Other choices of $k_i$, $g_2'$ and $\tilde{g}_2'$ will also have a $\theta$ dependent mass. Clearly this is not physically credible, and so the Lagrangian (66), despite being the most obvious generalisation of the $SU(5)$ GUT, is unrealistic.

### 6.2.5   Low Temperature $Y$-String Non-Neutrino Zero Modes

The $Y$-string can be approached in a similar way to the $W$-string, although the resulting equations are more complicated. Choosing a gauge in which $T_s = \frac{1}{2\sqrt{2}}(\tau^{Y_1'^+} + \tau^{Y_1'^-})$, the energetically favourable choice of $R$ is $s\frac{1}{2n}\tau^{Y_1^3}$, with $s = \pm 1$. $\tau^{Y_1^3}$ is the generator that forms an $SU(2)$ subgroup with $\tau^{Y_1^\pm}$. $\Phi_{10}$ is

then equal to

$$
\frac{\eta_2}{2\sqrt{2}}\left[\kappa(H^0 + s\tilde{H}^1)e^{isp\theta} + \tilde{\kappa}(\tilde{H}^0 + sH^1)e^{-isp\theta}\right]h_p
$$
$$
+\frac{\eta_2}{2\sqrt{2}}\left[\kappa(H^0 - s\tilde{H}^1)e^{-isq\theta} + \tilde{\kappa}(\tilde{H}^0 - sH^1)e^{isq\theta}\right]h_q
\tag{68}
$$

The problem is best expressed in terms of fermion eigenstates of $R$ and $T_s$. Putting $\Psi_L = \hat{u}'v + s\hat{d}'w + \frac{1}{\sqrt{2}}(\hat{u}'^c + s\hat{d}'^c)\chi^{(+)} + \frac{1}{\sqrt{2}}(\hat{u}'^c - s\hat{d}'^c)\chi^{(-)}$, where $\hat{u}' = a_1\hat{u}_1 + a_2\hat{u}_2^c + a_3\hat{u}_3^c$ and $\hat{d}' = a_1\hat{e}^+ + a_2\hat{d}_3 - a_3\hat{d}_2$ with $a_i$ arbitrary and real, gives

$$
e^{i\theta}\left(\partial_r + \frac{i}{r}\partial_\theta - \frac{sb(r)}{2r}\right)v_U + \frac{m_2}{\sqrt{2}}\tilde{\kappa}\left(h_pe^{-isp\theta}\chi_U^{(-)*} + h_qe^{isq\theta}\chi_U^{(+)*}\right) = 0 \tag{69}
$$

$$
e^{i\theta}\left(\partial_r + \frac{i}{r}\partial_\theta + \frac{sb(r)}{2r}\right)w_U + \frac{m_2}{\sqrt{2}}\kappa\left(h_pe^{isp\theta}\chi_U^{(+)*} - h_qe^{-isq\theta}\chi_U^{(-)*}\right) = 0 \tag{70}
$$

$$
e^{i\theta}\left(\partial_r + \frac{i}{r}\partial_\theta + \frac{sna(r)}{2r}\right)\chi_U^{(+)} + \frac{m_2}{\sqrt{2}}\left(\kappa h_pe^{isp\theta}w_U^* + \tilde{\kappa}h_qe^{isq\theta}v_U^*\right) = 0 \tag{71}
$$

$$
e^{i\theta}\left(\partial_r + \frac{i}{r}\partial_\theta - \frac{sna(r)}{2r}\right)\chi_U^{(-)} + \frac{m_2}{\sqrt{2}}\left(\tilde{\kappa}h_pe^{-isp\theta}v_U^* + \kappa h_qe^{-isq\theta}w_U^*\right) = 0 \tag{72}
$$

These equations are similar to (62) and (63). Their $\theta$ dependence is removed with the substitutions

$$
\begin{aligned}
v_U &= Ae^{il\theta} & w_U &= Be^{i(sq-sp+l)\theta} \\
\chi_U^{(+)} &= C^*e^{-i(1-sp+l)\theta} & \chi_U^{(-)} &= D^*e^{-i(sq+1+l)\theta}
\end{aligned}
\tag{73}
$$

The resulting equations have the same form as the corresponding $W$-string equations, and can be solved in the same way. This gives a total of $12q = 6(n-1)$ (or $6n$ if $n$ even) normalisable zero modes.

### 6.2.6  Low Temperature $W$ and $Y$ String Neutrino Zero Modes

For the fields affected by both $\Phi_{126}$ and $\Phi_{10}$ in the presence of a $W$-string, the equations of motion are a combination of (56) and (63,63)

$$
\begin{aligned}
e^{i\theta}&\left(\partial_r + \frac{i}{r}\partial_\theta \pm \frac{na(r)}{2r}\right)\chi_U^{c(\pm)} \\
&+m_1\tfrac{1}{2}\left(f_0(r)\chi_U^{c(\mp)*} + e^{\pm in\theta}f_1(r)\chi_U^{c(\pm)*}\right) \\
&+\frac{\kappa+\tilde{\kappa}}{2}m_2h_qe^{\pm iq\theta}\chi_U^{(\pm)*} - \frac{\kappa-\tilde{\kappa}}{2}m_2h_pe^{\pm ip\theta}\chi_U^{(\mp)*} = 0
\end{aligned}
\tag{74}
$$

$$
\begin{aligned}
e^{i\theta}&\left(\partial_r + \frac{i}{r}\partial_\theta \mp \frac{b(r)}{2r}\right)\chi_U^{(\pm)} \\
&+\frac{\kappa+\tilde{\kappa}}{2}m_2h_qe^{\pm iq\theta}\chi_U^{c(\pm)*} - \frac{\kappa-\tilde{\kappa}}{2}m_2h_pe^{\mp ip\theta}\chi_U^{c(\mp)*} = 0
\end{aligned}
\tag{75}
$$

The angular dependence is removed with the substitutions

$$
\begin{aligned}
\chi_U^{(+)} &= Ae^{il\theta} + E^*e^{-i(p-q+1+l)\theta} & \chi_U^{(-)} &= Be^{i(p-q+l)\theta} + F^*e^{-i(1+l)\theta} \\
\chi_U^{c(+)} &= C^*e^{-i(1-q+l)\theta} + Ge^{i(p+l)\theta} & \chi_U^{c(-)} &= D^*e^{-i(1+p+l)\theta} + He^{i(l-q)\theta}
\end{aligned} \tag{76}
$$

Only 4 of the 8 solutions are well behaved at large $r$. At small $r$, no more than 4 solutions are well behaved for any choice of $l$. Thus it is not possible to match up the different solutions to give one which is normalisable everywhere. This is also true for the $Y$-string, so neither of them have low temperature zero modes involving the conjugate neutrino field.

## 6.3  Summary

The only fermion zero modes that form at high temperatures are $\nu^c$ zero modes around abelian strings (in which case there are $|n|$ of them), or those that involve fermion fields that just couple to the string gauge fields, and not $\Phi_{126}$. This latter type of zero mode will only occur for higher $n$ strings ($|n| \geq 3$).

At low temperatures there are a total of $16|m|$ different zero modes on an abelian string ($|m|$ for each particle type), where $m$ is the winding number of $\Phi_{10}$. $m = 0$ when $|n| < 3$, so there are no zero modes around topologically stable abelian strings, and hence they can only be superconducting at low temperatures in the presence of an unusual Higgs potential [11].

If $m \neq 0$, and $z$ and time dependence is added to the solutions, they will correspond to superconducting fermion currents. The electron and down quark currents will then flow in the opposite direction to the neutrino and up quark currents.

In the presence of a $X'$ or $X_S$ nonabelian string there are no zero modes at any temperature. The other types of nonabelian string each have $12q$ zero modes ($q$ for each particle type not coupling to $\Phi_{126}$), where $q$ is the winding number of the part of $\Phi_{10}$ which winds least, so $q = \frac{1}{2}n$ for even $n$, and $q = \frac{1}{2}(n-1)$ for odd $n$. For a minimal energy, topologically stable string there are no fermion zero modes, although there is still the possibility of superconductivity due to gauge boson zero modes. Thus even the $X$ and $X_S$-strings may be superconducting [17]. Indeed, it has recently been shown that such strings do become current carrying by gauge boson condensation [20].

The supercurrents corresponding to any fermion zero modes present do not consist of single particle types, as those around an abelian string do. Instead they are made up of eigenstates of the string generator. Also, unlike the abelian case, currents containing each particle type flow in both directions along the string.

# 7  Other Related Grand Unified Theories

Although only one particular $SO(10)$ GUT has been discussed, many of the results apply to different symmetry breakings. Any theory of the form

$$SO(10)\cdots \xrightarrow{\Phi_{126}} \cdots SU(3)_c \times SU(2)_L \times U(1)_Y \times Z_2 \xrightarrow{\Phi_{10}} SU(3)_c \times U(1)_Q \times Z_2 \quad (77)$$

could have string solutions of the form (7) or (8), which would cause electroweak symmetry restoration at low temperatures in the same way as (1). The form of the other Higgs fields will not make much difference, as long as they are single valued in the presence of a string (like $\Phi_{45}$). If they are, it will not be necessary to add extra gauge fields, and so $\Phi_{10}$ will have the same behaviour as in (1). The resulting strings will have the same kind of zero modes as the (1) theory, provided none of the other Higgs fields couple to the fermions. The only Higgs fields that can couple to fermions are those which transform under a representation contained in the $\mathbf{16} \times \mathbf{16}$, since fermion mass terms transform as a product of $\mathbf{16}$s. The only such representations are $\mathbf{126}$, $\mathbf{10}$, and $\mathbf{120}$ (which is antisymmetric), so the results of the previous section apply to a wide range of theories.

The Higgs fields which gain their VEVs after $\Phi_{126}$ will determine the most energetically favourable choice of string generator, as $\Phi_{45}$ did in (1). If a GUT of the form (77) has Higgs fields which take non-zero VEVs before $\Phi_{126}$, the choice of $T_s$ will be more restricted. If a generator has already been broken, the formation of the corresponding string will not occur.

One theory of the form (77) is

$$
\begin{aligned}
SO(10) \quad &\xrightarrow{\Phi_A} \quad SU(5) \times U(1)_P \xrightarrow{\Phi_{45}} SU(3)_c \times SU(2)_L \times U(1)_Y \times U(1)_P \\
&\xrightarrow{\Phi_{126}} \quad SU(3)_c \times SU(2)_L \times U(1)_Y \times Z_2 \\
&\xrightarrow{\Phi_{10}} \quad SU(3)_c \times U(1)_Q \times Z_2
\end{aligned}
$$

$$(78)$$

The $\Phi_A$ Higgs field transforms under either the $\mathbf{45}$ or $\mathbf{210}$ representation of $SO(10)$, and is an $SU(5)$ singlet. Unlike (1), only abelian strings can form in this theory, since the only generator that $\Phi_{126}$ breaks is $P$. This means that electroweak symmetry restoration will always occur in the presence of a string. $\Phi_A$ and $\Phi_{45}$ will both take their usual VEVs in the presence of such a string, so the only symmetries restored in the string core will be $U(1)_P$, and the electroweak symmetry. Another interesting feature of this theory is that strings can form at energies close to the electroweak scale, and so $r_s$ could be of similar size to $r_{ew}$, although still smaller. This string is a candidate for defect mediated electroweak baryogenesis [6]. Switching the second and third symmetry breakings also gives a theory with similar solutions.

A different unifying gauge group (instead of $SO(10)$), with similar properties to (78) is $SU(5) \times U(1)_P$. It was suggested in [11], and has two independent gauge coupling constants. Unlike (78), strings of all winding numbers will be

topologically stable, since $U(1)_P$ is broken to $Z$ instead of $Z_2$. They could still decay by splitting into several strings with lower winding numbers. The field equations for the electroweak fields will be the same as (16) and (17), but with $-\frac{1}{5}$ the ratio of the two couplings instead of just $-\frac{1}{5}$. If the ratio is $\alpha$, then $\Phi_{10}$'s winding number will be the nearest integer to $-\alpha\frac{n}{5}$ (with half integers rounded towards zero), so if $|\alpha| > \frac{5}{2}$, $\Phi_{10}$ will always wind in the presence of a string. Since $|m|$ will always be non-zero, fermion zero modes will always be present. This also means that neutrino zero modes can survive the electroweak phase transition

A theory which is substantially different from (1) starts with the symmetry breaking $SO(10) \xrightarrow{\Phi_{54}} SU(4) \times SU(2)^2 \times Z_2^C$. The $Z_2^C$ symmetry is not the $Z_2$ symmetry in (77). $\Phi_{10}$ is not invariant under it, so it must be broken during or before the electroweak symmetry breaking. This will lead to formation of domain walls, and so such a theory will have substantially different properties to (1), and is ruled out cosmologically [1].

Another type of theory closely related to (77) occurs when $\Phi_{126}$ is replaced by $\Phi_{16}$, where the usual VEV of $\Phi_{16}$ is proportional to $u$. The gauge fields all gain masses in the same way as the equivalent theory involving $\Phi_{126}$, but there will be no discrete $Z_2$ symmetry, so there will be no topological strings. However, solutions of the form (7,8) can still form, although since $e^{2\pi i n T_s}$ will need to map $u$ to $u$ to give a single valued $\Phi_{16}$, only solutions with even $n$ will occur. Of course, if such strings are to be observed, they will need to be stable, which will only happen for certain values of the theory's parameters. Embedded defects similar to these have been discussed previously [15, 21].

Since $\Phi_{16}$ does not couple to the fermions, $g_1$ will be zero in (34), and so the neutrinos will have the same kind of zero modes as all the other particles (As would be the case if $\Phi_{126}$ were present, but $g_1$ were zero). However such a theory has left-handed neutrinos with significant masses, and so is not compatible with the standard model (unless some other mechanism is introduced to reduce the mass of the $\nu$ field).

Yet another set of related theories can be obtained from (1) by choosing a different VEV of $\Phi_{45}$. Adding a multiple of $P$ to it will not affect any of the $SU(5)$ symmetry breaking since all the $SU(5)$ fields commute with it, thus it will not alter which gauge bosons become superheavy. It will alter the sizes of the masses of the $SO(10)$ fields. The most energetically favoured choice of nonabelian string will be the one with the lowest energy contribution at the $\Phi_{45}$ symmetry breaking. This will be the one whose string generator corresponds to gauge fields with the lowest mass. So by choosing $\Phi_{45}$ appropriately, any of the nonabelian strings could become favourable. The $X'$ and $Y$ strings are gauge equivalent at this stage, but since the $X'$ string contributes nothing at the electroweak symmetry breaking, it will always be more favourable than the $Y$ string. Thus any of the $X'$, $X_S$ or $W$-strings could be energetically favourable. If it is the $W$-string, then it is most probable that electroweak

symmetry restoration will occur. The same sort of freedom does not exist with $\Phi_{126}$ and $\Phi_{10}$, since any such change will give different fermion mass terms, and radically alter the theory.

# 8    Conclusions

In this paper we have uncovered a very rich microstructure for $SO(10)$ cosmic strings. In particular, we have found four nonabelian strings as well as one abelian string. We have examined the effect of the strings on the subsequent symmetry breakings and studied the zero modes in detail. Our results are summarised in the table

| Gauge field | Type | Symmetry restoration | | | EW fields | Zero modes if | |
|---|---|---|---|---|---|---|---|
| | | $SO(10)$ | $SU(5)$ | EW | | GUT | EW |
| $S$ | abelian | yes | no | yes | $Z$ | $n \neq 0$ | $\lvert n \rvert \geq 3$ |
| $X_{Si}^{\pm}$ | nonabelian | partial | partial | no | — | $n \geq 3$ | never |
| $X_i'^{\pm}$ | nonabelian | partial | partial | no | — | $n \geq 3$ | never |
| $W_R^{\pm}$ | nonabelian | partial | partial | yes | $W_L^{\pm}$ | $n \geq 3$ | $n \geq 2$ |
| $Y_i'^{\pm}$ | nonabelian | partial | partial | yes | $A, Z$, gluons | $n \geq 3$ | $n \geq 2$ |

It seems that electroweak symmetry restoration by GUT strings is quite likely. The exact results are dependent on the details of the theory and the choice of string generator. For the $SO(10)$ theory considered, electroweak symmetry is restored for the abelian string, and half the possible nonabelian strings, although the most energetically favourable of these does not restore electroweak symmetry. However, other closely related $SO(10)$ GUTs have different minimal energy string solutions, which will restore the symmetry, such as (78), or (1) with a different choice of $\Phi_{45}$. Thus, our results generalise to a range of theories. This is currently under investigation.

The size of the region of electroweak symmetry restoration for the topologically stable ($n = 1$) strings is determined by the electroweak scale, and is much larger than the string core. For nonabelian strings, with higher winding number, the region will be the same if they are topologically equivalent to the $n = 1$ string (i.e. odd $n$), and restricted to the string core if they are topologically equivalent to the vacuum (i.e. even $n$). There is no such distinction between topological and non-topological abelian strings, which restore symmetry in the larger region if the winding number is not a multiple of 5. Some of the $SU(5)$ symmetry is also restored by all of the nonabelian strings, but not the abelian string. This is only within the string core, irrespective of the string winding number, and since $\Phi_{45}$ is not zero there, the restoration is only partial.

For any choice of $T_s$, ignoring possible potential driven symmetry restoration, the GUT will not be fully restored at the string's centre. All the nonabelian strings have non-zero (although smaller than usual) $\Phi_{126}$ and $\Phi_{45}$ fields at their

centre, so the $SO(10)$ (apart from the electroweak fields) symmetry is only partially restored inside the string. For the abelian string $\Phi_{126}$ is zero in the string core, but $\Phi_{45}$ takes its usual value, so with the exception of $U(1)_P$, most of the $SO(10)$ symmetry is broken. The resulting gauge boson masses are smaller, but still superheavy. However, for the abelian and the $W$ and $Y$ nonabelian strings, there is almost full restoration of electroweak symmetry in a larger region than the string core.

Although the profile of the electroweak Higgs field obeys the same boundary conditions as a string, its exact form has a closer resemblance to a string with non-integer winding number. For the abelian string the actual winding number of $\Phi_{10}$ is less than that of the GUT string (about $\frac{1}{5}$). The same is true for the nonabelian string, which has the winding number (or numbers if $n$ is odd) of $\Phi_{10}$ about $\frac{1}{2}$ that of the string itself.

For the abelian string it is the winding number of $\Phi_{10}$ that determines the existence of fermion zero modes after electroweak symmetry breaking. The number of zero modes is 16 times its winding number, so unfortunately there will be none for topologically stable strings, which have $|n| = 1$ and hence $m = 0$. Neutrino zero modes can always exist at high temperatures, but they do not survive the electroweak phase transition (for $|n| = 1$). This result is fairly general, as discussed in section 7.

The cosmology of such strings is rather interesting. The existence of neutrino zero modes at high temperatures enables the string to carry a neutral current, and thus lead to the formation of vortons [22]. Normally, vortons formed at such high temperatures result in the theory being ruled out cosmologically [22, 23]. However, in our case the vortons would cease to be stable below the electroweak scale, and cannot be used to rule out the theory [24].

At high temperatures, it is also possible (for higher $n$) for zero modes to form because of the string gauge fields. This can also occur for the nonabelian strings, and for the non-conjugate neutrino fields around abelian strings. However this effect is always overridden if the fermion field couples to a non-zero Higgs field.

In the presence of a nonabelian string, different parts of $\Phi_{10}$ can have different winding numbers. In the cases considered it is the part with the lowest winding number which determines the number of zero modes. Its winding number is equal to $\frac{1}{2}n$ rounded down to the nearest whole integer for $W$ and $Y$ strings, and 0 for the other two types. There are then a total of 12 times this number of possible fermion zero modes. The fields coupling to $\Phi_{126}$ (part of which has winding number 0) do not have such solutions. As with the abelian string there are no zero energy fermion solutions for topologically stable strings, and so fermion zero modes on strings are not as common as would be expected.

In our analysis we have only considered terms occuring in the tree level Lagrangian. One-loop corrections are likely to induce couplings between the nonabelian string field and the electroweak Higgs. This may result in electroweak symmetry restoration around the $X'$ and $X_S$ strings. However, the electroweak Higgs field would not wind in this region, and there would still be no fermion

zero modes.

# A  $SO(10)$ **Grand Unified Theory**

Under $SO(10)$, all left-handed fermions transform under one representation, and right-handed fermions transform under its conjugate [18]. It is convienient to use just one representation. This can be achieved by using the charge conjugates of the fermions ($\psi_R^c = C\bar{\psi}_R^T$, $\psi_L^c = C\bar{\psi}_L^T$). The charge conjugate of a right handed fermion transforms as a left handed fermion, and vice versa. Thus $\Psi_L = \psi_L + \psi_R^c$ is left handed. For $SO(10)$ this definition is necessary, as well as convienient. $\psi_L$ could be gauge transformed to $\psi_R^c$, so any gauge invariant quantities will have to involve just $\Psi_L$ and $\Psi_R$ (right-handed equivalent of $\Psi_L$). However, $\Psi_R$ is superfluous, since it is equal to $i\sigma_2 \Psi_L^* = \Psi_L^c$, so the theory can be described entirely in terms of $\Psi_L$. For the electron family, it can be written as

$$\Psi_L^{(e)} = \left(u_1, u_2, u_3, \nu_e, d_1, d_2, d_3, e^-, d_1^c, d_2^c, d_3^c, e^+, u_1^c, u_2^c, u_3^c, \nu_e^c\right)^T \tag{79}$$

where $d_i = d_{iL}$, $d_i^c = i\sigma_2 d_{iR}^*$, etc. so all the fields are left handed. The other two families of fermions can be described similarly. Adapting work by Rajpoot [19], the gauge fields can be expressed explicitly as $16 \times 16$ matrices

$$A = \sqrt{2} \begin{pmatrix} H & I_4 W_L^- & M_Y' & M_X \\ I_4 W_L^+ & H & M_X'^* & M_Y \\ M_Y'^\dagger & M_X'^T & -H^* & I_4 W_R^- \\ M_X^\dagger & M_Y^\dagger & I_4 W_R^+ & -H^* \end{pmatrix} + \Lambda \tag{80}$$

where $I_4$ is the $4 \times 4$ identity matrix, and

$$M_X = \begin{pmatrix} 0 & -X_3^- & X_2^- & -X_1'^- \\ X_3^- & 0 & -X_1^- & -X_2'^- \\ -X_2^- & X_1^- & 0 & -X_3'^- \\ X_1'^- & X_2'^- & X_3'^- & 0 \end{pmatrix} \tag{81}$$

$$M_Y = \begin{pmatrix} 0 & Y_3^- & -Y_2^- & -Y_1'^+ \\ -Y_3^- & 0 & Y_1^- & -Y_2'^+ \\ Y_2^- & -Y_1^- & 0 & -Y_3'^+ \\ Y_1'^+ & Y_2'^+ & Y_3'^+ & 0 \end{pmatrix} \tag{82}$$

30

$M'_X$ and $M'_Y$ are obtained by swapping $X_i^{\pm}$ and $Y_i^{\pm}$ with $X_i'^{\pm}$ and $Y_i'^{\pm}$ in $M_X$ and $M_Y$. $H$ is defined as

$$
H = \begin{pmatrix} & \mathbf{G} & & X_{S1}^- \\ & & & X_{S2}^- \\ & & & X_{S3}^- \\ X_{S1}^+ & X_{S2}^+ & X_{S3}^+ & 0 \end{pmatrix}
\tag{83}
$$

$G$ is a $3 \times 3$ matrix of containing the gluon fields, it is hermitian, and so $H$ is too. The other fields are contained in the diagonal matrix $\Lambda$

$$
\begin{aligned}
\Lambda &= \mathrm{diag}\Big( (\tfrac{B'}{\sqrt{6}} + W_L^3)_3, -3\tfrac{B'}{\sqrt{6}} + W_L^3, (\tfrac{B'}{\sqrt{6}} - W_L^3)_3, -3\tfrac{B'}{\sqrt{6}} - W_L^3, \\
&\qquad (-\tfrac{B'}{\sqrt{6}} + W_R^3)_3, 3\tfrac{B'}{\sqrt{6}} + W_R^3, (-\tfrac{B'}{\sqrt{6}} - W_R^3)_3, 3\tfrac{B'}{\sqrt{6}} - W_R^3 \Big) \\
&= \mathrm{diag}((s + 2a + 2z)_3, -3s + 4z, (s - 3z - a)_3, -3s - 3a - z, \\
&\qquad (-3s + a - z)_3, s + 3a - 3z, (s - 2a + 2z)_3, 5s)
\end{aligned}
\tag{84}
$$

The subscripts indicate repeated values, and

$$
\begin{aligned}
s &= \tfrac{1}{5}\left(-W_R^3 + \sqrt{\tfrac{3}{2}}B'\right) \\
B &= \sqrt{\tfrac{3}{5}}W_R^3 + \sqrt{\tfrac{2}{5}}B' \\
z &= \tfrac{1}{4}\left(W_L^3 - \sqrt{\tfrac{3}{5}}B\right) \\
a &= \tfrac{1}{4}\left(W_L^3 + \sqrt{\tfrac{5}{3}}B\right)
\end{aligned}
\tag{85}
$$

$Z = \sqrt{10}z$ and $A = \sqrt{6}a$ are the unrenormalised electroweak $Z^0$ boson and photon respectively. $S = -\sqrt{10}s$ is a high energy $SO(10)$ boson. The generator $P$ is obtained by putting $s = 1$ and $a = z = 0$ in the expression for $\Lambda$. The substitutions $s = z = 0$ and $a = \tfrac{1}{3}$ give the charge operator.

## A.1   Higgs Fields

The electroweak Higgs field transforms under the $\mathbf{5}_{-2}$ and $\bar{\mathbf{5}}_2$ representations of $SU(5)$. They are contained in the $\mathbf{10}$ of $SO(10)$, and so the components of $\Phi_{10}$ can be expressed as symmetric products of spinors transforming under the $\mathbf{16}$ representation. Thus $\Phi_{10}$ can be expressed as $\phi_\alpha H^\alpha + \tilde{\phi}_\alpha \tilde{H}^\alpha$, where $H^\alpha$ ($\alpha = 0, +, 1, 2, 3$), are the five components of $\mathbf{5}_{-2}$, and $\tilde{H}^\alpha$ are the corresponding components of $\bar{\mathbf{5}}_2$. $(H^0, H^+)$ and $(\tilde{H}^0, \tilde{H}^+)$ form $SU(2)_L$ doublets, while $H^i$ and $\tilde{H}^i$ ($i = 1, 2, 3$) form an $SU(3)_c$ triplet and antitriplet. Expressing these components of $\mathbf{10}$ in terms of symmetric products of $\mathbf{16}$s gives

$$
\begin{aligned}
H^0 &= \tfrac{1}{4}[(\hat{d}_j \times \hat{d}_j^c)_S + (\hat{e}^+ \times \hat{e}^-)_S] & H^+ &= \tfrac{1}{4}[(\hat{u}_j \times \hat{d}_j^c)_S + (\hat{e}^+ \times \hat{\nu})_S] \\
H^i &= \tfrac{1}{4}[\epsilon_{ijk}(\hat{d}_j^c \times \hat{u}_k^c)_S - (\hat{u}_i \times \hat{e}^-)_S + (\hat{d}_i \times \hat{\nu})_S] \\
\tilde{H}^0 &= \tfrac{1}{4}[(\hat{u}_j \times \hat{u}_j^c)_S + (\hat{\nu} \times \hat{\nu}^c)_S] & \tilde{H}^+ &= \tfrac{1}{4}[(\hat{u}_j^c \times \hat{d}_j)_S + (\hat{e}^- \times \hat{\nu}^c)_S] \\
\tilde{H}^i &= \tfrac{1}{4}[\epsilon_{ijk}(\hat{u}_j \times \hat{d}_k)_S - (\hat{d}_i^c \times \hat{\nu}^c)_S + (\hat{u}_i^c \times \hat{e}^+)_S]
\end{aligned}
\tag{86}
$$

where $\hat{\nu}$ is the basis vector corresponding the $\nu$ field, etc. The second Higgs field $\Phi_{45}$ is in the $\mathbf{24}_0$ component of the $\mathbf{45}$ representation, and its usual vacuum expectation value is proportional to the diagonal matrix

$$\text{diag}\left((1/3)_3, -1, (1/3)_3, -1, (2/3)_3, 2, (-4/3)_3, 0\right) \tag{87}$$

In the absence of a string, $\Phi_{126}$ is proportional to $(u \times u)_S$, with $u = \hat{\nu}^c$.

## A.2 Fermion Masses

The masses of the fermions arise from Yukawa couplings to the Higgs fields. These must of course be Lorentz and gauge invariant. The two possible Lorentz invariant mass terms are Dirac masses ($\bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L$), and Majorana masses ($\bar{\psi}_L^c \psi_L + \bar{\psi}_L \psi_L^c +$ (right-hand terms)). Dirac masses transform as singlets under gauge transformations, and so will not be gauge invariant when coupled to a Higgs field. The Majorana masses transform as a product of $\mathbf{16}$s, and so can be coupled to similarly transforming Higgs fields, $\Phi_{126}$ and $\Phi_{10}$, but not $\Phi_{45}$. This gives the fermionic Lagrangian (34), (considering only one family of fermions for simplicity), with Yukawa couplings.

Of course, since (34) is invariant under $SO(10)$, it must be invariant under $SU(5)$ as well. Thus $\Phi_{126}$ can only couple to $SU(5)$ singlets (i.e. products of the conjugate neutrino field). The two components of $\Phi_{10}$ couple to $\mathbf{5}_{-2}$ and $\bar{\mathbf{5}}_2$ products of fermions. Under $SU(5)$, the remaining fermions transform under $\mathbf{10}_1$ and $\bar{\mathbf{5}}_{-3}$ representations. To find allowable mass terms, products of these representations need to be expressed in terms of irreducible representations. $\mathbf{10}_1 \times \mathbf{10}_1$ and $\mathbf{1}_5 \times \bar{\mathbf{5}}_{-3}$ both contain $\bar{\mathbf{5}}_2$s, and $\mathbf{10}_1 \times \bar{\mathbf{5}}_{-3}$ contains a $\mathbf{5}_{-2}$. Thus, for the usual VEVs of the Higgs fields, the mass terms are written in terms of particle fields as

$$\begin{aligned} \bar{\Psi}_L H^0 \Psi_L^c &= \tfrac{1}{8}\left[d_i^\dagger i\sigma_2 d_i^{c*} + e^{-\dagger} i\sigma_2 e^{+*} + d_i^{c\dagger} i\sigma_2 d_i^* + e^{+\dagger} i\sigma_2 e^{-*}\right] \\ &= \tfrac{1}{4}\left[\bar{d}_{iL} d_{iR} + \bar{e}_L^- e_R^-\right] \end{aligned} \tag{88}$$

and similarly for $\tilde{H}^0$, so

$$\bar{\Psi}_L \Phi_{10} \Psi_L^c = \frac{\eta_2 \kappa}{4\sqrt{2}}\left[\bar{d}_{iL} d_{iR} + \bar{e}_L^- e_R^-\right] + \frac{\eta_2 \tilde{\kappa}}{4\sqrt{2}}\left[\bar{u}_{iL} u_{iR} + \bar{\nu}_L \nu_R\right] \tag{89}$$

$$\bar{\Psi}_L \Phi_{126} \Psi_L^c = \nu^{c\dagger} i\sigma_2 \nu^{c*} \frac{\eta_1}{\sqrt{2}} = \nu_R^T i\sigma_2 \nu_R \frac{\eta_1}{\sqrt{2}} \tag{90}$$

This model, unlike the standard model, has non-zero neutrino masses. However, if the $\nu_R^T i\sigma_2 \nu_R$ term is much larger than the $\bar{\nu}_L \nu_R$ term, the mass eigenstates will be approximately $\nu_L$ and $\nu_R$, and have very small and very large mass eigenvalues respectively, giving an almost massless left-handed neutrino.

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